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A MATHEMATICAL EXAMINATION OF THE PRESS MODEL FOR ATMOSPHERIC TURBULENCE

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16. Abstract

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SUMMARY

The random process used to model atmospheric turbulence in aircraft response problems is examined on a mathematical basis. The first, second, and higher order probability density and characteristic functions are developed. The concepts of the Press model lead to an approximate procedure for the analysis of the response of linear dynamic systems to a class of non-Gaussian random processes. The Press model accounts for both the Gaussian and non-Gaussian forms of measured turbulence data. The nonstationary aspects of measured data are explicitly described by the transition properties of the random process. The effects of the distribution of the intensity process upon calculated exceedances are examined. It is concluded that the Press model with a Gaussian intensity distribution gives a conservative prediction of limit load values.

INTRODUCTION

The random process developed by Press and his associates at NASA (refs. 1 to 3) is used to model the random nature of atmospheric turbulence in many aeronautical applications. The process is an application of random process theory to the problems of aircraft response to atmospheric turbulence. It was developed to account for the exceedance properties of atmospheric tubulence indicated by data from VGH recorders on many aircraft. The turbulence was considered to consist of a series of local regions or "patches," each having a stationary Gaussian distribution of turbulence velocity components with a fixed intensity or standard deviation. The intensity was then varied by a separate random process in order to account for the apparent nonstationary and non-Gaussian structure of the atmospheric turbulence. A schematic example of the pattern of locally Gaussian regions with intensity variations between the regions is shown in figure 1.

The probabilistic structure of the resulting random process was first examined by Bullen. (See refs. 4 and 5.) Other studies of the random process are the reports by Dutton and Thompson (ref. 6) and by Reeves (ref. 7). The application of the process to

*This research was accomplished while the author held a National Research Council Postdoctoral Resident Research Associateship at NASA Langley Research Center.

atmospheric turbulence data has also been examined by Coupry (refs. 8 and 9) who referred to the development as the Press process, a terminology which is followed in the present report. The associated exponential type of exceedance expression for the average rate of exceeding a level of aircraft response has been the basis for the evaluation of an extensive amount of experimental data. Some of this work has been discussed in references 10 to 13. Several studies have developed the associated analytical techniques (the "power spectral density method") for application in aircraft strength design. (See refs. 14 to 16.) These procedures are included in some versions of aircraft strength criteria. (See refs. 17 and 18.)

The present study examines the mathematical basis of the Press process and examines several applications both to the measurement of atmospheric turbulence and to the associated problems of aircraft response. The primary conclusion is that the basic concepts of the Press process, which were originally developed largely on an intuitive basis, can be expressed with an explicit mathematical model. The study is divided into two parts. The first part is a general mathematical formulation of the Press process. Initially, the first order distributions of the process are developed, that is, distributions of an ensemble at a single time point. Then the second and higher order distributions are developed; this development leads to a discussion of the dynamics of the random process.

The second part of the study considers several applications of the Press process. Two specific mathematical problems are discussed. First, the validity of currently used analytical procedures as a method for the analysis of dynamic system response to a class of non-Gaussian random processes is discussed. Second, the ability of the Press process to account for the nonstationary behavior of atmospheric turbulence is discussed. It is also shown that the transition properties of the Press process explicitly account for both the Gaussian and non-Gaussian forms of experimental data. The use of the Press process as a specific model for atmospheric turbulence in aeronautical applications is then discussed. The effects of the choice of the intensity process upon both measured and calculated exceedance data are considered.

SYMBOLS

A	standard deviation of R process
b	standard deviation of S process
b_i	intensity parameter
$C(\ )$	characteristic function of subscripted process

det	determinant of a matrix
$\{dR(\ )\}$	spectral process associated with R process
$E[\ ]$	ensemble average
$F(\ )$	spectral distribution function of subscripted process
H	covariance matrix of vector R process
H_{ij}	elements of covariance matrix of vector R process
$h(\tau)$	impulsive response function of a linear system
i	unit of imaginaries, $\sqrt{-1}$
inv	inverse of a matrix
$K_n(\ )$	modified Bessel function of order n (ref. 22)
M_4	flatness factor (ratio of fourth moment to square of second moment)
m	dimension of vector process
$N(\ )$	expected rate of positive slope crossings of indicated level
N_o	expected rate of positive slope zero crossings, $\frac{1}{2\pi}\frac{A_{\dot{r}}}{A_r}$
n	index of generalized Press process (=1/2, 1, 3/2, . . .)
P_i	probability parameter
$p(\ )$	probability density function of subscripted process
$p_c(z\|s)$	conditional probability density function of z process conditional on value s
r,R	random process

s, S	random process ("intensity")
T	time interval
t	time
U_σ	turbulence or gust intensity parameter
x, X	quasi-steady random process (Press or generalized Press)
x_o	mean value of x
y, Y	generalized Press random process
z, Z	product or Press random process
α	$= \sqrt{1 - \rho^2}$
β	$Ab\theta$
$\Gamma(\ )$	gamma function (ref. 22)
δ_{ij}	Kronecker δ-symbol
θ	Fourier transform variable for characteristic functions
ρ	correlation coefficient, correlation function, or autocorrelation function
Σ, σ	random process
τ	time-difference variable
$\Phi(\omega)$	power spectral density function of subscripted process, $-\infty \leqq \omega \leqq \infty$
$\Psi(\ )$	autocovariance function of subscripted process
ω	frequency, Fourier transform variable of τ

$\langle\ \rangle$ time average

Superscript:

ℓ integer index

Subscripts:

c conditional

d derivative

i integer index

in input random process

j integer index

l limit

out output random process

tr transition

A bar over a symbol denotes a vector. Dots over symbols denote derivatives with respect to time. Primes denote derivatives with respect to the argument. A tilde over a symbol denotes a variable of integration.

FIRST ORDER DISTRIBUTIONS

The formulation of the Press process is developed in the present section. The development extends some previous work which has been done on the mathematical formulation of the Press process in references 4 and 6. The basis of the present development is the interpretation of the Press process as the product of two independent random processes, a concept first stated by Reeves (ref. 7).

The discussion is restricted to first order distributions; that is, the process is considered at a single time point only. Thus, the treatment of the time evolution of the random process is not considered in this section. The development follows standard analytical techniques of probability theory; appropriate references are the texts by

Papoulis (ref. 19) and by Lin (ref. 20). All the indicated statistical moments are assumed to exist. All the random processes are specified to have zero mean values.

Product Process

The random process is defined by the product of two component processes,

$$z(t_1) = r(t_1)\, s(t_1) \tag{1}$$

The two component processes, R and S, are specified to be statistically independent.

The probability functions of the product process Z are completely determined by the functions of the two component processes. The joint probability density function of the product process and one of its components is determined by the product relation (eq. (1)) and the independence of the two component processes

$$p_{zs}(z,s) = \frac{1}{|s|}\, p_r(r = z/s) p_s(s) \tag{2}$$

The probability density function of the product process itself is obtained from the joint density function by using the consistency relation, which determines the marginal distribution from the joint probability density function

$$p_z(z) = \int_{-\infty}^{\infty} \frac{1}{|s|}\, p_r(r = z/s) p_s(s)\, ds \tag{3}$$

The R component process can be related to an associated conditional probability density function. By using the definition of conditional probability and the form of the joint probability density function (eq. (2)),

$$p_c(z|s) = \frac{1}{|s|}\, p_r(r = z/s) \tag{4}$$

The probability density function of the product process can be written in terms of the conditional probability density function as

$$p_z(z) = \int_{-\infty}^{\infty} p_c(z|s)\, p_s(s)\, ds \tag{5}$$

The probability density function of the product process can be developed from either equation (3) or equation (5), depending upon whether the R component process is specified either directly or indirectly by the associated conditional process. The historical approach (ref. 1) used the idea of the conditional probability and developed equation (4) through the specific form assumed for the conditional process. The interpretation of the resulting random process as a product process was introduced by Reeves (ref. 7). The product relation (eq. (1)) is particularly useful in the subsequent development since it leads directly to some of the important properties of the Press process and its extensions.

The product process can also be developed in terms of its characteristic function

$$C_z(\theta) = \int_{-\infty}^{\infty} e^{i\theta z}\, p_z(z)\, dz \tag{6}$$

The characteristic function is developed from equation (3) by using the properties of the associated Fourier transformations (ref. 21)

$$C_z(\theta) = \int_{-\infty}^{\infty} C_r(\theta s)\, p_s(s)\, ds \tag{7}$$

The formulation of the product process in terms of characteristic functions is convenient since it usually leads to simpler functional relations.

The statistical moments of the product process can be developed from those of the component processes; either from both the probability density and the characteristic functions or directly from the product relation and the independence of the component processes

$$E\left[z^{\ell}(t_1)\right] = E\left[r^{\ell}(t_1)\right] E\left[s^{\ell}(t_1)\right] \tag{8}$$

The preceding development considers the product process in general terms. The subsequent development is restricted to the specific forms of the two component processes which are useful in applications as a model for atmospheric turbulence.

Press Process

The original development (ref. 1) of the Press random process specified the R process to be Gaussian under the locally Gaussian condition. All the present development is restricted to this case. The reason for this choice is discussed later in the section on the quasi-steady approximation. Several distributions were considered for the S process in reference 1. A Gaussian distribution was selected, based on the correspondence between the resulting exponential exceedance expression and measured atmospheric turbulence data. Almost all the subsequent applications of the Press model have been based upon the S component being a Gaussian random process. This particular case is referred to as the Press process.

Scalar-scalar.- The Press process in scalar form is generated by the product of two scalar Gaussian independent component processes, R and S.

$$\left.\begin{aligned} E\left[r^2(t_1)\right] &= A^2 \\ E\left[s^2(t_1)\right] &= b^2 \end{aligned}\right\} \tag{9}$$

The statistical moments of the Press process can be determined directly from the product relation and the independence of the R and S process.

$$\left.\begin{aligned} E\left[z^\ell(t_1)\right] &= 0 && (\ell,\ \text{odd}) \\ E\left[z^\ell(t_1)\right] &= \left[1 \cdot 3 \cdot 5 \ .\ .\ .\ (\ell - 1)\right]^2 A^\ell b^\ell && (\ell,\ \text{even}) \end{aligned}\right\} \tag{10}$$

The probability density functions of the component processes have the Gaussian functional form:

$$\left.\begin{aligned} p_r(r) &= \frac{1}{\sqrt{2\pi}A} \exp\left(\frac{-r^2}{2A^2}\right) && (-\infty \leqq r \leqq \infty) \\ p_s(s) &= \frac{1}{\sqrt{2\pi}b} \exp\left(\frac{-s^2}{2b^2}\right) && (-\infty \leqq s \leqq \infty) \end{aligned}\right\} \tag{11}$$

This notation is fairly standard in the aeronautical literature, except that the S component process is usually restricted to nonnegative values. However, the S process must be defined with both positive and negative values in order to avoid some mathematical problems in the subsequent formulation. This approach does not lead to any incompatibility with the traditional approach to the Press model. The possible restriction to nonnegative values is based upon the historical development of the Press process through the concept of a conditional process, using equations (4) and (11), where the conditional standard deviation is the product of the standard deviation of the R process and the modulus of the value of the S process. In order to satisfy this interpretation, a separate random process termed Σ, which is restricted to nonnegative values, is defined. The value of the Σ process is the modulus of the value of the S process. The relations between these two processes are developed in appendix A.

The probability density function of the Press process is determined by equation (3) and by the functions of the two component processes

$$p_z(z) = \frac{1}{\pi Ab} K_0\left(\frac{|z|}{Ab}\right) \qquad (-\infty \leqq z \leqq \infty) \tag{12}$$

This relation was first derived by Bullen (ref. 4). The indicated function is a modified Bessel function of the second kind and zero order. (See ref. 22.) The corresponding characteristic function is obtained either from the component processes by using equation (7) or from the Fourier transformation of the probability density function (eq. (12)).

$$C_z(\theta) = \left(A^2 b^2 \theta^2 + 1\right)^{-1/2} \tag{13}$$

By using the asymptotic expansion for the modified Bessel function, the probability density function of the Press process has basically an exponential form for large values of the argument

$$p_z(|z| >> bA) \rightarrow \frac{1}{\sqrt{2\pi Ab|z|}} \exp\left(\frac{-|z|}{Ab}\right) \tag{14}$$

The strongly non-Gaussian nature of the Press process is evident from the exponential-type dependence of the probability density function in its extremes. Another measure of the Gaussian character is the flatness factor, the ratio of the fourth moment to the square of the second moment.

$$M_4 = \frac{E\left[z^4(t_1)\right]}{\left\{E\left[z^2(t_1)\right]\right\}^2} \tag{15}$$

The moments are given by equations (10). The flatness factor has a value of nine for the Press process in contrast to the value of three for the Gaussian process. Thus, the first order probability density function and the flatness factor both indicate the strongly non-Gaussian nature of the Press process.

Vector-scalar.- The development is easily extended to a vector random process. However, the vector process can be defined in several ways, depending upon whether the R process or the S process or both are considered to be vector processes. The present development is guided by previous applications of the Press process in the aeronautical literature, which consider the vector nature of the process to be due solely to the R process

$$z_j(t_1) = r_j(t_1)\ s(t_1) \tag{16}$$

Thus, the vector process Z is formed by the product of a vector process R, whose components correspond to those of the Z process, and a scalar process S. The distribution of the R process is specified to be jointly Gaussian with appropriate covariance matrix

$$E\left[r_i(t_1)\ r_j(t_1)\right] = H_{ij} \tag{17}$$

The joint probability density and characteristic functions are developed from equations (3) and (7), with appropriate modifications for the vector nature of the R process. The probability density function of the vector-scalar Press process is

$$p_Z(\overline{z}) = \left[\frac{2}{\pi}\,\frac{1}{(2\pi)^m\ \det\,(H)}\right]^{1/2} \frac{1}{b^m}\left(\frac{D}{b}\right)^{\frac{1-m}{2}} K_{\frac{m-1}{2}}\left(\frac{D}{b}\right) \tag{18}$$

$$D^2 = \sum_{i=1}^{m}\sum_{j=1}^{m} z_i(\mathrm{inv}\ H)_{ij}\ z_j$$

The joint characteristic function is derived from the joint characteristic function of the R process by using the vector form of equation (7)

$$C_Z(\overline{\theta}) = \left(b^2 \sum_{i=1}^{m}\sum_{j=1}^{m} \theta_i H_{ij} \theta_j + 1\right)^{-1/2} \tag{19}$$

The quantity m is the dimension of the vector R and Z processes.

The various moments of the vector Press process can be obtained from the moments of the component processes by using the product relation and the independence of the R and S processes.

The joint probability density function of the Press process determines the probability density function of various combinations of the vector components. For an example, the sum of two components of a vector process is

$$z_{sum} = z_1 + z_2 = (r_1 + r_2)s = r_{sum}s \tag{20}$$

The random process R_{sum} is Gaussian since it is the sum of two Gaussian processes. Thus, the random process Z_{sum} is a Press process since it is the product of two independent Gaussian random processes. Therefore, the probability density of the sum is that of a scalar Press process with appropriate variance, determined from equation (20). This example shows that a linear combination of Press processes, all having the same scalar S process, is itself a Press process.

The vector Press process described is defined with a scalar S process. In this case the resulting components of the Press process are never independent. This property follows either from the probability density function, from the characteristic function, or directly from the product relation (eq. (16)).

Vector-vector.- A vector Press process can be defined in another manner as the product of a vector R and a vector S process:

$$z_j(t_1) = r_j(t_1)\ s_j(t_1) \tag{21}$$

In this case the probability density and characteristic functions are not those of equations (18) and (19). Also, in this case the components of the vector process can be independent.

Generalized Press Process

Distributions of the S process, other than the Gaussian, have been considered both in the original development (ref. 1) and in several other reports. An extensive list of possible S component processes and the resulting product processes has been compiled by Houbolt (ref. 23). In the present section the choice of the S process is generalized by consideration of a family of processes related to the Pearson type III family. This idea was originally suggested by Bullen (ref. 5), by Houbolt (ref. 23), and by Dutton and Thompson (ref. 6).

The S process is specified to be a member of a family of random processes related to the Pearson type III family

$$p_S(s;n) = \frac{1}{2^n\,\Gamma(n)}\,\frac{1}{|s|}\left(\frac{|s|}{b}\right)^{2n} \exp\left(\frac{-s^2}{2b^2}\right) \qquad (n = 1/2,\ 1,\ 3/2,\ .\ .\ .) \tag{22}$$

The Gaussian R component process is not changed. The resulting family of random processes is called a generalized Press process or simply a generalized process of index n. The family includes the special cases of the Gaussian (n = 1/2) and Rayleigh (n = 1) distributions of the S component process. Thus, the Press process discussed in the previous section is a generalized process of index one-half.

The probabilistic structure of the generalized process is determined from the relations for the R and S component process. The specific relations for the moments and for both the probability density and characteristic functions are developed in appendix B.

SECOND AND HIGHER ORDER DISTRIBUTIONS

The time evolution of the Press process is considered in the present section. The appropriate properties are defined by the higher order distributions: the joint probability of the random process considered at several time points. The second order distributions are most important since these are required for consideration of the derivative and the exceedance expression of the random process. Also, the second order distributions are related to the covariance and power spectral density functions. Both the R and S processes are specified to be stationary and to have zero mean values. All indicated derivatives of the random processes and all indicated statistical moments are assumed to exist.

Product Process

The product process is developed from the product of two independent scalar random processes.

$$z(t) = r(t)\ s(t) \tag{23}$$

The autocovariance function is the joint moment of the random process at two time points

$$E\left[z(t_1)\ z(t_2)\right] = \Psi_z\left(\left|t_2 - t_1\right|\right) = \Psi_z(\tau) \tag{24}$$

By using the product relation, the autocovariance of the product process is determined by the autocovariance functions of the R and S processes

$$\Psi_z(\tau) = \Psi_r(\tau)\ \Psi_s(\tau) \tag{25}$$

Since the three random processes are stationary, their autocovariance functions depend only upon the difference of the values of the two time points considered.

The derivative of the product process is directly related to the derivatives of the component processes

$$\dot{z}(t) = \dot{r}(t)\ s(t) + r(t)\ \dot{s}(t) \tag{26}$$

The autocovariance function of the derivative of the product process can be developed either from equation (25) or from equation (26) as

$$\Psi_{\dot{z}}(\tau) = -\Psi_z''(\tau) = \Psi_{\dot{r}}(\tau)\ \Psi_s(\tau) + \Psi_r(\tau)\ \Psi_{\dot{s}}(\tau) - 2\Psi_r'(\tau)\ \Psi_s'(\tau) \tag{27}$$

Thus, the autocovariance function of the derivative of the product process cannot be directly related to those of the derivatives of the two component processes alone; there is a mixed term not identified with the derivatives of either component process. This term vanishes in the special case of the variance of the derivative,

$$E\left[\dot{z}^2(t)\right] = \Psi_{\dot{z}}(0) = E\left[\dot{r}^2(t)\right]\ E\left[s^2(t)\right] + E\left[r^2(t)\right]\ E\left[\dot{s}^2(t)\right] \tag{28}$$

The autocovariance functions of higher order derivatives can be developed in a similar manner.

The relationship between the product and component processes can be developed in terms of their power spectral density functions. The three processes are assumed to be mean square continuous; thus, they can be expressed in a Fourier or spectral expansion. The associated power spectral density function is related to the autocovariance function by the Fourier transformation relations

$$\left.\begin{aligned} \Psi_z(\tau) &= \int_{-\infty}^{\infty} e^{i\omega\tau}\Phi_z(\omega)\ d\omega \\ \Phi_z(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\tau}\Psi_z(\tau)\ d\tau \end{aligned}\right\} \tag{29}$$

Corresponding relations can be written for the R and S processes, with appropriate changes in the notation.

The power spectral density function of the product process is determined by the corresponding functions of the component processes. The product relation between the autocovariance functions (eq. (25)) transforms into a convolution relation between the power spectral density functions

$$\Phi_z(\omega) = \int_{-\infty}^{\infty} \Phi_r(\omega - \tilde{\omega})\, \Phi_s(\tilde{\omega})\, d\tilde{\omega} \tag{30}$$

The relations between the autocorrelation and power spectral density functions have been given previously in reference 24.

The probability density functions of the product process are determined from the corresponding functions of the component processes. The second order probability density functions are considered, that is, the relations for the processes considered at two time points. The joint second order probability density function of the product process and one of its components is determined by the product relation (eq. (23)) and the independence of the two component processes as

$$p_{zs}(z_1,z_2;s_1,s_2) = \frac{1}{|s_1 s_2|}\, p_r(r_1,r_2)\, p_s(s_1,s_2) \tag{31}$$

The second order probability density function of the product process is obtained from the joint density function as

$$p_z(z_1,z_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{|s_1 s_2|}\, p_r(r_1,r_2)\, p_s(s_1,s_2)\, ds_1\, ds_2 \tag{32}$$

The corresponding second order characteristic function is obtained from the probability density function by using the properties of the associated Fourier transformations

$$C_z(\theta_1,\theta_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_r(\theta_1 s_1,\theta_2 s_2)\, p_s(s_1,s_2)\, ds_1\, ds_2 \tag{33}$$

The preceding three relations are similar to the corresponding first order relations (eqs. (2), (3), and (7)). The formulation can be extended to higher order distributions.

It is sometimes useful to interpret the formulation in terms of the associated conditional process. The appropriate relations are developed either from equation (31) or from equation (33) by using the definition of the conditional probability density function

$$p_c\left(z_1,z_2\middle|s_1,s_2\right) = \frac{1}{|s_1s_2|} p_r(r_1,r_2) \tag{34}$$

$$C_c\left(\theta_1,\theta_2\middle|s_1,s_2\right) = C_r(\theta_1 s_1,\theta_2 s_2) \tag{35}$$

The preceding development considers the product process in general terms. The subsequent development is restricted to specific forms of the two component processes.

Press Process

The Press process is a special case of the product process; both components, R and S, are specified to be stationary Gaussian processes. The scalar form of the Press process is considered first. The two scalar component processes are completely specified by their covariance functions and zero mean values. The second and higher order distributions of the Press process are completely determined by the covariance functions of the components. The covariance functions can be expressed in terms of the autocorrelation functions and the variances as

$$\left.\begin{aligned} E\left[r(t_1)\ r(t_2)\right] &= \Psi_r(\tau) = A^2\ \rho_r(\tau) \\ E\left[s(t_1)\ s(t_2)\right] &= \Psi_s(\tau) = b^2\ \rho_s(\tau) \\ E\left[z(t_1)\ z(t_2)\right] &= \Psi_z(\tau) = A^2b^2\ \rho_z(\tau) \end{aligned}\right\} \tag{36}$$

$$\rho_z(\tau) = \rho_r(\tau)\ \rho_s(\tau) \tag{37}$$

The relation between the autocorrelation functions follows from the relation between the autocovariance functions (eq. (25)).

Similar relations can be written for the first derivative of the random processes by using equation (27). The autocorrelation functions of the derivatives are determined from the autocorrelation functions of the random processes as

$$\rho_{\dot{z}}(\tau) = \frac{\rho_z''(\tau)}{\rho_z''(0)} \tag{38}$$

The second order Gaussian probability density and characteristic functions of the two component processes are

$$\left.\begin{aligned} p_r(r_1,r_2) &= \frac{1}{2\pi A^2\,\alpha_r(\tau)} \exp\left\{\frac{-1}{2A^2\alpha_r{}^2}\left[r_1{}^2 - 2\,\rho_r(\tau)\,r_1r_2 + r_2{}^2\right]\right\} \\ \alpha_r{}^2(\tau) &= 1 - \rho_r{}^2(\tau) \end{aligned}\right\} \tag{39}$$

$$C_r(\theta_1,\theta_2) = \exp\left\{-\frac{1}{2}\,A^2\left[\theta_1{}^2 + 2\,\rho_r(\tau)\,\theta_1\theta_2 + \theta_2{}^2\right]\right\} \tag{40}$$

Similar relations can be written for the S process with appropriate changes in notation as indicated by equation (36).

The second order probability density and characteristic functions of the Press process are obtained from the product relations (eqs. (31) to (35)) and the functional relations for the R and S component processes. However, in the subsequent analysis it is simpler to work with the characteristic functions. The second order characteristic function of the Press process is obtained by using equation (33)

$$\left.\begin{aligned} C_z(\theta_1,\theta_2) &= \left(\alpha_r{}^2\alpha_s{}^2\beta_1{}^2\beta_2{}^2 + \beta_1{}^2 + 2\rho_r\rho_s\beta_1\beta_2 + \beta_2{}^2 + 1\right)^{-1/2} \\ \beta_i &= Ab\theta_i \qquad\qquad (i = 1,\ 2) \end{aligned}\right\} \tag{41}$$

The characteristic function must be inverted in both transform variables to give the second order probability density function of the Press process. This inversion appears to be intractable. However, some of the important properties of the process can be obtained from the characteristic function. The second order characteristic function defines the corresponding moments of the Press process, although these are more easily obtained directly from the product relation. It is noted that the joint characteristic function depends only upon the difference of the values of the two time points, which is a consequence of the stationarity of the process.

The second order characteristic function can be used to determine several properties of the derivative of the Press process. The appropriate relations are given by Moyal (ref. 25). The characteristic function of the derivative is obtained directly from the second order characteristic function

$$\left.\begin{aligned} C_{\dot{z}}(\theta_d) &= \lim_{\tau \to 0} C_z\left(\frac{-\theta_d}{\tau}, \frac{\theta_d}{\tau}\right) = \left[1 - \beta_d^2 \rho_r''(0)\right]^{-1/2} \left[1 - \beta_d^2 \rho_s''(0)\right]^{-1/2} \\ \beta_d &= Ab\theta_d \end{aligned}\right\} \tag{42}$$

The characteristic function of the derivative is separated into two factors. Each factor is the characteristic function of a Press process by equation (13) and is dependent upon the derivative of only one of the component processes. By equation (26) the derivative is the sum of two processes, each of which is the product of two independent Gaussian processes. Thus, the derivative process is the sum of two independent Press processes, which is the statement of equations (42).

The joint characteristic function of the Press process and its first derivative is used to examine the independence of the random process and its derivative and to determine the exceedance expression. The joint characteristic function is determined from the second order function as

$$\left.\begin{aligned} C_{z\dot{z}}(\theta_z, \theta_d) &= \left\{\beta_z^2 + \left[1 - \beta_d^2 \rho_r''(0)\right]\left[1 - \beta_d^2 \rho_s''(0)\right]\right\}^{-1/2} \\ \beta_z &= Ab\theta_z \\ \beta_d &= Ab\theta_d \end{aligned}\right\} \tag{43}$$

Since the joint characteristic function does not separate into a product of two separate functions of the individual Fourier transform variables, the random process and its first derivative are not independent, although they are uncorrelated since the process is stationary.

The joint probability density function of the random process and its first derivative is obtained by inversion of the joint characteristic function in both variables. The joint density function is used to determine the expected rate of exceedances of a given value of the process, often referred to as the exceedance expression (ref. 26),

$$N(z) = \int_0^\infty \dot{z}\, p_{z\dot{z}}(z, \dot{z})\, d\dot{z} \tag{44}$$

Evaluation of both the joint probability density function and the exceedance expression from the joint characteristic function appears to be intractable.

The higher order distributions of the scalar random process are developed in a similar manner. Since both the R and S processes are Gaussian, their higher order probability density and characteristic functions are determined by the matrices of the autocovariance functions

$$\left.\begin{aligned} E\left[r(t_i)\ r(t_j)\right] &= A^2\ \rho_r(t_i - t_j) \\ E\left[r(t_i)\ r(t_j)\right] &= A^2 \rho_{r,ij} \end{aligned}\right\} \tag{45}$$

The matrix of the autocovariance function of the S process is defined by similar notation. The higher order characteristic functions of the Press process are determined by the same procedure as that used for the second order function (eqs. (41)). The appropriate relations are expressed in the forms for the higher order distribution of the scalar processes: the basic characteristic function relation (eq. (33)), the characteristic function of the R process, and the probability density function of the S process. The higher order characteristic functions of the Press process are then obtained by integrating all the elements of the S process in the relation corresponding to equation (33)

$$\left.\begin{aligned} C_{\overline{Z}}(\overline{\theta}) &= \left[\det\left(m_{ij} + \delta_{ij}\right)\right]^{-1/2} \\ m_{ij} &= A^2 b^2 \theta_i \sum_{\ell=1}^{m} \rho_{r,i\ell}\theta_\ell \rho_{s,\ell j} \end{aligned}\right\} \tag{46}$$

The scalar Press process is formally identical to a vector Press process at a single time point if the time evolution of the S component is suppressed. The higher order characteristic function of a scalar Press process (eqs. (46)) can be reduced to the first order characteristic function of a vector Press process (eq. (19)). This relationship is shown by giving a unit value to all elements of ρ_s and by interpreting the values of the scalar R process at given time points as the components of a vector R process at a single time point. This relationship can be extended to develop the higher order characteristic functions of a vector process. These functions are obtained from equations (46) by replacing the variances and correlation matrices with the appropriate covariance matrices of the vector R and of the scalar or vector S component processes.

Generalized Press Process

The generalized Press process is formed by the product of the Gaussian R process with a S process which is a member of a family of processes related to the Pearson type III family. The first order probability density functions of the S process family are given by equation (22). The second order functions of the generalized process are determined directly from the second order functions of the two component processes and the appropriate product relations (eqs. (32) or (33)). However, the higher order functions are more easily obtained in a different manner by using the relation between the generalized processes of various indices, that is, the generalized process of index 2n is the sum of n independent and identically distributed Press processes. This relation (eq. (B9)) determines the higher order functions of the generalized process from those of the Press process itself. This approach is discussed further in appendix B.

In summary, the probabilistic functions of the Press and the generalized Press processes are developed by using the representation of the Press process as the product of two independent random processes. The probabilistic structure of the Press process is completely determined from the given properties of the component processes. The characteristic functions of arbitrary order can be developed for both the scalar and vector Press processes by introducing the appropriate notation in equations (46).

APPLICATIONS OF PRESS PROCESS

The mathematical properties of the Press process are developed in the previous sections. The present section examines the relationship between the mathematical properties and the intuitive concepts of the Press process as originally developed and as applied both to atmospheric turbulence measurement problems and to associated aircraft response problems. The intent of the present section is to show how the intuitive concepts are related to and largely justified by the mathematical development. Two general topics are considered. The first topic is the development of the approximate form of the random process which is used for the analysis of the dynamic response of linear systems. This approximation is based upon the concept of the S process being slowly varying so that it affects system response only in a static manner. This concept is termed the quasi-steady approximation. The second topic is the transition properties of the Press process and their relation to the stationarity and the Gaussian character of data modeled by the Press process.

The Quasi-Steady Approximation

The quasi-steady approximation of the Press process is discussed in the present section. The quasi-steady form of the process is developed from the exact random process. The validity of the quasi-steady approximation as a method for the analysis of dynamic system response to a class of non-Gaussian random processes is discussed.

The time variation of the Press process requires consideration of the time variations of both the R and the S processes. The dynamics of the exact Press process are influenced by the dynamics of both component processes. However, the historical development (refs. 1 and 3) introduced a different concept which is an approximation of the exact process dynamics. This concept is based on the turbulence patch idea; the turbulence field consists of patches where the random turbulence velocity component (Z) varies with negligible change of the random intensity or magnitude (related to S). This concept introduces the idea of two time scales. On a local time scale the Gaussian R process varies at a constant value of the S process. On a global time scale the S process varies, and introduces the non-Gaussian nature of the product process. This concept is called the quasi-steady approximation. The corresponding random process is called the quasi-steady Press process or simply the quasi-steady process.

The development and the limitations of the quasi-steady approximation are shown explicitly by the consideration of the response of a linear dynamic system. The system response or output is the convolution of the impulsive response of the system and the system input (refs. 19 and 20).

$$z_{out}(t) = \int_0^\infty h(\tau)\, z_{in}(t - \tau)\, d\tau \tag{47}$$

The input Press process is the product of two independent component processes (eq. (23)) so that equation (47) can be written as

$$z_{out}(t) = \int_0^\infty h(\tau)\, r_{in}(t - \tau)\, s_{in}(t - \tau)\, d\tau \tag{48}$$

It is assumed that the impulsive response of the system is essentially restricted to some time period T. In other words if the impulsive response is set to zero after a time period T, the change in the system response calculated by equation (47) is negligible. With this restriction the integration in equation (48) can be limited to the system response time period.

$$z_{out}(t) \approx \int_0^T h(\tau)\, r_{in}(t - \tau)\, s_{in}(t - \tau)\, d\tau \tag{49}$$

At this point the quasi-steady concept is introduced; the S component process is essentially constant over the system response time period T. With this assumption the S component can be removed from the integration since it is independent of the integration variable.

$$z_{out}(t) \approx s_{in}(t) \int_0^T h(\tau)\ r_{in}(t - \tau)\ d\tau \tag{50}$$

The integration now involves only the R component process. The integral is identified as the response of the system to the R component process.

$$r_{out}(t) = \int_0^T h(\tau)\ r_{in}(t - \tau)\ d\tau \tag{51}$$

Since the input R component process is Gaussian, then the output R component process also is since it is a linear combination of Gaussian random variables. The system response is the product of two independent component processes and combines equations (50) and (51)

$$z_{out}(t) \approx r_{out}(t)\ s_{in}(t) \tag{52}$$

Thus, the probabilistic structure of the system output is completely determined within the accuracy of the quasi-steady approximation. Further, the system output and input have the same probabilistic structure since they are the product of a Gaussian R process and the same S process.

The development of the system response (eq. (52)) shows the structure and assumptions of the quasi-steady process. The R component process is always specified to be Gaussian. There is no restriction on the distribution of the S component process since this component is not affected by the system response. It is noted that this pattern of the restrictions on the two-component process is precisely the pattern used in the original development. (See refs. 1 and 3.) The time variation of the S component process is assumed to be much slower than all significant aspects of the system response. In applications to atmospheric turbulence problems, the R component process is also assumed to vary on a much smaller time scale than the S component. This is the "locally Gaussian" condition. The development of equation (52) shows that this assumption is actually not necessary. However, this assumption is followed in the subsequent development, since the case of a rapidly varying R component is of primary importance in dynamic response analysis.

The quasi-steady approximation presents a method of analysis of the response of dynamic systems to one class of strongly non-Gaussian processes. In applications the input R component process is defined to have unit variance; that is,

$$E\left[r_{in}^2(t)\right] = 1 \tag{53}$$

This definition establishes an arbitrary scale factor between the two component processes. The system response is determined by the response R component process. For linear systems, the response is completely specified in a probabilistic sense by the covariance function of the Gaussian R component and the variance of the S component process.

The probabilistic relations for the quasi-steady process are obtained from those for the exact Press process. Either of two analytical procedures can be used. In the first procedure the relations for the quasi-steady process are obtained by developing the dynamics of the Gaussian R process, and then introducing the quasi-steady S process by means of consistency relation. This method is the standard approach used in the development and application of the Press process in the aeronautical literature. This procedure is the same as that for developing the first order relations of a vector Press process formed by the product of a vector R and a scalar S component (eqs. (18) and (19)). In the second procedure the relations for the quasi-steady process are obtained directly from the corresponding relations for the exact process by setting the local variation of the S process to zero. The second method explicitly shows the development of the quasi-steady approximation.

An example of the quasi-steady approximation is the development of the derivative of the product process (eq. (26))

$$\dot{z}(t) = \dot{r}(t)\ s(t) + r(t)\ \dot{s}(t)$$

The relative contributions of the two component processes to the variance of the derivative are expressed in terms of the derivatives of the two autocorrelation functions.

$$E\left[\dot{z}^2(t)\right] = A^2 b^2\left[-\rho_r''(0) - \rho_s''(0)\right] \tag{54}$$

Under the quasi-steady approximation, the time variation of the S process is negligible in comparison with that of the R process. The autocorrelation function of the S component is constant on a local time scale. The variance of the derivative of the quasi-steady process is

$$E\left[\dot{x}^2(t)\right] = -A^2 b^2 \rho_r''(0) \tag{55}$$

(The notation of the Z and X processes is used to distinguish the exact and quasi-steady processes, respectively.)

Similarly, the characteristic function of the derivative of the quasi-steady process is developed from that of the exact Press process given by equations (42)

$$C_{\dot{Z}}(\theta) = \left[1 - A^2 b^2 \theta^2 \rho_r''(0)\right]^{-1/2} \left[1 - A^2 b^2 \theta^2 \rho_s''(0)\right]^{-1/2}$$

$$C_{\dot{X}}(\theta) = \left[1 - A^2 b^2 \theta^2 \rho_r''(0)\right]^{-1/2} \tag{56}$$

Thus, the derivative of the exact process is the sum of two independent Press processes. The derivative of the quasi-steady process is itself a Press process. This relation is equivalent to obtaining the derivative of the quasi-steady process from the product relation by omitting the derivative of the S component

$$\dot{x}(t) = \dot{r}(t)\ s(t) \tag{57}$$

One important application of the quasi-steady concept is the development of the associated exceedance expression, which is the expected rate of crossings of a given level of a random process. The exceedance expression is obtained from the joint probability density of the random process and its first derivative (ref. 26)

$$N(x) = \int_0^{\infty} \dot{x}\ p_{X\dot{X}}(x,\dot{x})\ d\dot{x} \tag{58}$$

An alternate approach is to use the exceedance expression of the R process which is Gaussian. The exceedance expression of the total process is obtained by using the consistency relation to eliminate the S process

$$N(x) = \int_0^{\infty} N(x|s)\ p_S(s)\ ds \tag{59}$$

The exceedance expression for the quasi-steady Press process is determined by either of these approaches. Historically, the development of the exceedance expression has followed equation (59)

$$N(x) = N_0 \exp\left(-\frac{|x|}{Ab}\right) \tag{60}$$

The exceedance expression for the quasi-steady generalized Press process of index n is obtained in a similar manner since only the distribution of the S process is changed

$$N(x;n) = \frac{N_o}{2^{n-1}\,\Gamma(n)}\left(\frac{|x|}{Ab}\right)^n K_n\left(\frac{|x|}{Ab}\right) \tag{61}$$

The quasi-steady process is an approximate form of the exact Press process. The time variation of the S component process is assumed to be much slower than both the R component process and all significant aspects of the dynamic response of a system. The quasi-steady assumption is an approximate method for the analysis of the response of linear dynamic system to a class of strongly non-Gaussian random processes.

Transition Properties

The transition properties of the Press process are examined in the present section. By using the transition properties, the relation between the properties of the Press process and the properties of stationarity and Gaussianness is also examined.

Measured exceedance data for atmospheric turbulence velocity components show a dual nature, that is, both Gaussian and exponential-type distributions. The Gaussian distribution appears in some of the direct measurements of atmospheric turbulence, generally those related to short periods of turbulence. The exponential-type distribution also appears in some of the direct turbulence measurements and is particularly valid for airplane VGH data measured over a long period of time. (See refs. 12 and 27.) A discussion of the existence of both types of data is given in reference 28.

The original development of the Press process (ref. 1) attempted to account for this dual nature by introducing the turbulence patch concept. Thus, data measured within one patch are Gaussian; data measured over a large number of patches have an exponential-type distribution. The transition between the two distributions is interpreted as an indication of nonstationarity.

The questions of the stationarity and the Gaussian character of the Press process are formally answered by the functional form of the probability density functions; the Press process is stationary and non-Gaussian. The formal definition of these properties does not answer the questions regarding experimental data. However, these questions can be restated in a more specific form. First, how can the Press process, which is stationary, account for data which appear to be nonstationary? Second, how can the Press process, which is non-Gaussian, account for data which appear to be Gaussian? The answers to both of these questions are contained in the concept of the product of two independent random processes, which evolve on radically different time scales.

The transition effects of a random process are described by a *transition probability density function*, the probability density function of a random process conditional on its value at an earlier time point. Any measurement of a random function implicitly involves consideration of a transition probability density function which determines the transition of the process from a given initial state to a random final state. In other words, a set of measured data, started at some point in time, must forget its initial value to some extent. With the usual type of random process the transition effects are eliminated by measuring data over a time period which is many times greater than the time scale of the random process. With the Press process this operation is complicated by the presence of two independent random processes that have radically different time scales. Thus, the transition process occurs in two steps and is indicated schematically by the following notation:

$$
\begin{aligned}
& p\left[z(t_2)\,\middle|\,r(t_1), s(t_1)\right] \\
& \quad \rightarrow p\left[z(t_2)\,\middle|\,s(t_1)\right] \\
& \quad\quad \rightarrow p\left[z(t_2)\right] \qquad\qquad (t_1 < t_2)
\end{aligned}
$$

The first transition involves the loss of the initial value of the R process. For the case of atmospheric turbulence, the time scale of this process is usually on the order of a few seconds, which is usually much less than the measurement period. Thus, this transition is of secondary importance in most applications; it is omitted in the subsequent discussion. The second transition, involving the loss of the initial value of the S process, may involve a time scale on the order of the measurement period. This transition may strongly influence the properties of the measured data.

The important measured quantity is the conditional or transitional probability density function of the random process at time t_2, given the value of the S process at an earlier time t_1. This function is determined from the consistency relation and the definition of conditional probability as

$$
\left.
\begin{aligned}
p(z_2|s_1) &= \int_{-\infty}^{\infty} p(z_2|s_1,s_2)\, p(s_2|s_1)\, ds_2 \\
p(z_2|s_1) &= \int_{-\infty}^{\infty} p(z_2|s_2)\, p(s_2|s_1)\, ds_2
\end{aligned}
\right\} \tag{62}
$$

The second relation follows from a special functional property of the conditional probability density function

$$p(z_2|s_1,s_2) = p(z_2|s_2) \tag{63}$$

This relationship is proven by using the properties of the associated characteristic function

$$\left.\begin{aligned} C(\theta_2|s_1,s_2) &= C_c(\theta_1 = 0,\theta_2|s_1,s_2) \\ C(\theta_2|s_1,s_2) &= C_r(\theta_1 s_1 = 0,\theta_2 s_2) \end{aligned}\right\} \tag{64}$$

The first equality is the consistency relation for the characteristic function which obtains the marginal function from the second order characteristic function. The second equality, which is a functional property (eq. (35)) of the second order conditional characteristic function, shows that the indicated characteristic function is independent of $s(t_1)$.

The relation (eq. (63)) does not imply an independence property between the product process Z and the S component process.

The two probability density functions under the integral sign in equations (62) are both Gaussian for the Press process. The indicated integration cannot be performed directly but requires a series expansion and integration term by term.

$$p(z_2|s_1) = \frac{1}{\pi A b \alpha_s} \exp\left(-\frac{1}{2}\lambda^2\right) \sum_{\ell=0}^{\infty} \frac{1}{(2\ell)!}\left(\lambda^2 \frac{\zeta}{\alpha_s}\right)^{\ell} K_\ell\left(\frac{\zeta}{\alpha_s}\right) \tag{65}$$

$$\lambda = \frac{\rho_s}{\alpha_s}\frac{s_1}{b}$$

$$\alpha_s = \sqrt{1 - \rho_s^2}$$

$$\zeta = \frac{|z_2|}{Ab} \neq 0$$

The convergence of the series follows from comparison with the multiplication theorem of Bessel functions (ref. 22). The series converges for all finite values of the parameter λ. Accordingly, the series converges for all values of the modulus of $\rho_s(\tau)$ less than one.

The transitional probability density function (eq. (65)) defines the transition of the Press process from the initial random state to the final random state. The initial and final states are determined by considering the limiting values of $\rho_s(\tau)$ as the time difference becomes zero and infinity, respectively,

$$\lim_{\rho_s(\tau \to 0) \to 1} p(z_2|s_1) = \frac{1}{\sqrt{2\pi} A |s_1|} \exp\left(\frac{-z_2^2}{2A^2 s_1^2}\right) \tag{66}$$

$$\lim_{\rho_s(\tau \to \infty) \to 0} p(z_2|s_1) = \frac{1}{\pi Ab} K_0\left[\frac{|z_2|}{Ab}\right] \tag{67}$$

Equation (66) follows from the integral expression (eqs. (62)). The development requires consideration of the Dirac delta functional form of the transition probability density function of the S process in the limit of small time differences. Equation (66) does not follow from the series expression (eq. (65)), since that series is not uniformly convergent. Equation (67) follows either from the integral expression or from the series expression.

The transition probability density function is Gaussian only in the limit of zero time difference; the Press process itself is non-Gaussian. Also, the transition probability density function depends only upon the time difference. This property is a direct consequence of the stationarity of the Press process, although the transition itself is nonstationary.

The transition properties of the Press process can be developed in terms of an associated transition random process. This process is defined by the transition probability density function (eqs. (62)). Comparison of equations (62) and (5) shows that the transitional process is the product of two independent random processes, the fully developed R process and the transitonal S process

$$z_{tr}(t) = r(t)\, s_{tr}(t) \tag{68}$$

The transitional S component process is defined by the probability density function of $s(t_2)$ conditional on the value of $s(t_1)$. It is Gaussian with known mean and variance, both dependent on $\rho_s(\tau)$.

The transition process generates an associated set of transition moments

$$E\left(z_{tr}^{\ell}\right) = \int_{-\infty}^{\infty} z_2^{\ell}\, p\left(z_2 \middle| s_1\right) dz_2 \tag{69}$$

These moments can be calculated directly from the product relation (eq. (68))

$$\left.\begin{aligned} E\left(z_{tr}^{\ell}\right) &= E\left(r^{\ell}\right) E\left(s_{tr}^{\ell}\right) \\ E\left(s_{tr}^{\ell}\right) &= \int_{-\infty}^{\infty} s_2^{\ell}\, p\left(s_2 \middle| s_1\right) ds_2 \end{aligned}\right\} \tag{70}$$

The transition moments are defined by ensemble averages only, since the transition process is nonstationary. The relation between the ensemble averages and the associated time averages is discussed subsequently.

The moments show the transition of the Press process in a concise form. The second and fourth moments are of primary interest:

$$\left.\begin{aligned} E\left(z_{tr}^{2}\right) &= A^2\left[b^2 + \rho_s^{\,2}(\tau)\left(s_1^{\,2} - b^2\right)\right] \\ E\left(z_{tr}^{4}\right) &= 3A^4\left[3b^4 + 6\,\rho_s^{\,2}(\tau)\left(s_1^{\,2} - b^2\right) + \rho_s^{\,4}(\tau)\left(s_1^{\,4} - 6s_1^{\,2}b^2 + 3b^4\right)\right] \end{aligned}\right\} \tag{71}$$

The moments show the transition through the autocorrelation function of the S process. The transition variance shows the development of the variance of the process from its given initial value to its final value which corresponds to the fully developed Press process

$$E\left(z_{tr}^{2}\right) = \begin{cases} A^2 s_1^{\,2} & (\tau \to 0) \\ A^2 b^2 & (\tau \to \infty) \end{cases} \tag{72}$$

The transition of the probabilistic structure is also shown by the corresponding flatness factor

$$M_{4,tr}(\tau) = \frac{E\left(z_{tr}^{4}\right)}{\left[E\left(z_{tr}^{2}\right)\right]^2} \tag{73}$$

The transition flatness factor is plotted in figure 2 as a function of the autocorrelation function of the S process. The transition flatness factor varies from the value of three for the initial Gaussian distribution to the value of nine for the fully developed Press distribution. The development of the transition process is strongly influenced by the transition of the variance of the S process, from its given initial value to the variance of the fully developed process. The transition of the Press process is slower in terms of ρ_S for higher initial values of the S process.

The development of transition moments can be extended to the case of the generalized Press process. The moments are developed from the corresponding transition process, which is generated in the same manner as the generalized process itself. Accordingly, the transition process is the sum of mutually independent and identically distributed transition Press processes, defined by a relation similar to equation (B9).

$$\left.\begin{aligned} y_{tr}(t;n) &= \sum_{j=1}^{2n} \left[z_{tr}(t)\right]_j \\ y_{tr}(t;n) &= \sum_{j=1}^{2n} r_j(s_{tr})_j \end{aligned}\right\} \tag{74}$$

The transition moments of the generalized process can be expressed in terms of the moments of the Press process itself, which is a generalized process of index n equal to one-half

$$\left.\begin{aligned} E\left(y_{tr}^2\right) &= 2n\ E\left(z_{tr}^2\right) \\ E\left(y_{tr}^4\right) &= 2n\ E\left(z_{tr}^4\right) + 6n(2n - 1)\left[E\left(z_{tr}^2\right)\right]^2 \end{aligned}\right\} \tag{75}$$

In a similar manner the transition flatness factor of the generalized process is related to that of the Press process

$$M_{4,tr}(\tau;n) = \frac{1}{2n}\left[M_{4,tr}(\tau;n = 1/2) - 3\right] + 3 \tag{76}$$

The transition flatness factors for the generalized processes are plotted in figure 3 as a function of the autocorrelation function of the S process. The transition flatness factor varies from the value of three for the initial Gaussian distribution to the value for the fully developed process given by equation (B6). The transition factors also show the trend toward a Gaussian distribution as the index of the generalized process is increased. Only the special case where the initial value is equal to the standard deviation of the S process is considered; the additional effect of the transition of the variance (eqs. (75)) is not shown.

The transition properties of the Press process can also be examined by means of the exceedance expression, which is a commonly measured quantity in atmospheric turbulence applications. The quasi-steady approximation is used to simplify the development. The exceedance expression for the transition process is developed in the same manner as the expression for the fully developed process (eq. (59))

$$N(x_2|s_1) = \int_{-\infty}^{\infty} N(x_2|s_2)\, p(s_2|s_1)\, ds_2 \tag{77}$$

The transition exceedance expression is the expected rate of exceedances of a specified level at time t_2 given the value of the S process at an earlier time t_1. The procedure for the integration of equation (77) is similar to that for equations (62)

$$N(x_2|s_1) = N_0\sqrt{\frac{2\zeta}{\pi\alpha_s}}\, \exp\left(-\frac{1}{2}\lambda^2\right) \sum_{\ell=0}^{\infty} \frac{1}{(2\ell)!}\left(\lambda^2 \frac{\zeta}{\alpha_s}\right)^{\ell} K_{\ell+\frac{1}{2}}\left(\frac{\zeta}{\alpha_s}\right) \tag{78}$$

The notation follows that of equation (65).

An example of the transition exceedance expression is shown in figure 4. The exceedance expression shows the development of the process from the initial Gaussian distribution through intermediate states to the final exponential form of the fully developed Press process. The expected rate of exceedances is strongly affected by the transition, particularly for large values of the random process. The transition exceedance expression can have a predominantly exponential form when ρ_s is still fairly large. The example of figure 4 is the special case where the given initial value of the S process is equal to its standard deviation. Thus, the example shows the transition of the probabilistic structure but does not show the additional effect of the transition of the variance. Figure 5 is an example of the transition exceedance expression which shows the effects of the transition of both the probabilistic structure and the variance. In this case the development of the variance increases the transition effects upon the exceedance expression, particularly for large values of the random process.

In the preceding development the statistical moments of the transition process are defined by ensemble averages. Since the transition process is nonstationary, the ensemble averages are generally not equal to the time averages of a single record of the process. For a stationary and ergodic process, the two types of averages can be equated with reasonable accuracy if the period of time measurement is sufficiently large. This relation can possibly be applied to the transition process since that process is stationary on a local time scale. Thus, the ensemble averages of the transition process can be replaced by time averages if the variation of the S process is negligible both with respect to the development of the R process and with respect to the required time measurement period of the R process. Both of these conditions are contained in the concept of the quasi-steady process.

The measurement of experimental data over a sufficiently long period of time is influenced by the development of the S process. The transition exceedance expression developed previously is essentially an instantaneous quantity, although the expression includes the full development of the R process. The time-averaged exceedance expression is related to the instantaneous quantity defined previously,

$$\left\langle N\left(x \middle| s_1\right)\right\rangle = \frac{1}{T}\int_{t_1}^{t_1+T} N\left[x(\tilde{t})\middle| s(t_1)\right]\, d\tilde{t} \tag{79}$$

The time-averaged expression depends on the length of the measurement period. The calculation of the averaged exceedance function from the instantaneous function (eq. (79)) requires knowledge of the time dependence of the transition process, which is completely determined by the autocorrelation function of the S process.

Transition properties are generated by the evolution of a random process. Transition properties are possessed by all random processes, excluding the case of a totally static process. For the usual random process these properties represent the transition from a discrete distribution at a given initial value to the continuous distribution of the fully developed process. In the Press process the development occurs in two stages because of the different time scales of the two component processes. The first stage represents the development of the R process with negligible development of the S process. The second stage, which is considered in the previous discussion, represents the development of the S process. During this stage, the probability density function develops between two continuous distributions rather than the usual transition from a discrete to a continuous distribution.

The Press process has a formal simularity to the analytical model of evolutionary spectra used for the analysis of nonstationary processes. The concepts of evolutionary spectra and oscillatory processes have been developed by Priestley (ref. 29). These concepts have been applied to the problem of aircraft response to atmospheric turbulence by Howell and Lin (ref. 30), and Verdon and Steiner (ref. 31). The relationship between the Press and the oscillatory processes is discussed in appendix C.

In summary, the properties of the Press process answer several questions regarding the stationarity and the Gaussian character of measured atmospheric turbulence data. The answers to these questions are contained in the concept of the product of two independent random processes which evolve on radically different time scales. The transition properties of the Press process generate a comprehensive model of the nonstationary aspects of atmospheric turbulence. The transition properties of the Press process account for both the Gaussian and the exponential forms of measured data and explicitly account for the transition between these two forms.

APPLICATIONS TO ATMOSPHERIC TURBULENCE PROBLEMS

The application of the Press and the generalized Press processes to atmospheric turbulence problems is considered in this section. Two basic subjects are considered: first, the application of the processes to measured turbulence data and second, the effect of the distribution of the S component process upon calculated exceedances. A specific example of an airplane mission analysis is developed. The relation between the distribution of the S process and the rate of the exceedance of the extreme loads associated with the determination of structural strength is examined.

Measured Atmospheric Turbulence Data

The application of the Press process to atmospheric turbulence measurements requires the introduction of the concept of several types of turbulence. This concept was introduced in the original development (ref. 1) and has extensive experimental justification. The S component process is replaced by a conditional process, which is conditional on the type of turbulence. The probability density function of the modified S process is obtained by the consistency relation

$$p_S(s) = \sum_i P_i \, p(s|i) \tag{80}$$

The parameters P_i are the probabilities of occurrence of the ith type of turbulence and are generally referred to as the probability parameters. Each of the conditional probability density functions (eq. (80)) has a separate standard deviation, whose associated parameters b_i are generally referred to as the intensity parameters.

Most developments consider two types of turbulence, that is, two terms in the summation in equation (80). These are generally referred to as nonstorm (i = 1) and storm (i = 2) turbulence, although the specific identification is questionable. There have been suggestions for a third term to represent extreme turbulence. There is an additional term, omitted in the literature, which represents the probability of no turbulence. This term does not contribute to any of the measured quantities except for the zero values of the random process. However, this term is necessary to insure that the total probability is equal to one.

The modified Press process is formed by the product of the Gaussian R process and the modified S process. For the modified Press process each conditional distribution (eq. (80)) is Gaussian. The resulting S component process is not Gaussian if there is more than one type of turbulence, although it is usually identified as Gaussian in the aeronautical literature. The probability density function and the exceedance expression for the modified Press process are obtained from the relations for the component processes and equations (3) and (59)

$$p_x(x) = \frac{1}{\pi A} \sum_{i=1}^{2} \frac{P_i}{b_i} K_o\left(\frac{|x - x_o|}{Ab_i}\right) \tag{81}$$

$$\frac{N(x)}{N_o} = \sum_{i=1}^{2} P_i \exp\left(\frac{-|x - x_o|}{Ab_i}\right) \tag{82}$$

The mean value of the random process is included in the preceding expressions. The exceedance expression (eq. (82)) has been the basis of extensive experimental study of atmospheric turbulence, for example, references 11 to 16. A review of the experimental determination of the associated atmospheric parameters is given in reference 12.

The procedure can be extended to the generalized Press process by using the relations for the generalized process development in appendix B.

$$\frac{N(x;n)}{N_o} = \frac{1}{2^{n-1}\,\Gamma(n)} \sum_{i=1}^{2} P_i\left(\frac{|x - x_o|}{Ab_i}\right)^n K_n\left(\frac{|x - x_o|}{Ab_i}\right) \tag{83}$$

The Press process has an inherent dependence property which affects different components both of a vector process and of quantities formed from higher order distributions. This property is caused by different quantities being generated with a common S component process. The dependence property is not changed either by the specific form of the S process or by the quasi-steady approximation. This property introduces some subtleties in the properties of the Press process which are important in applications to atmospheric turbulence. Two examples are considered.

The Press process and its first derivative are not independent. They are uncorrelated since the process is stationary. However, the independence property is absent since the two quantities have the same S component process. One consequence of this property is that the exceedance expression and the probability density have different functional forms (eqs. (81) and (82)). This is consistent with the development of the exceedance expression from the joint probability density function of the process and its first derivative (ref. 26). Consequently, it is necessary to consider the exceedance and the probability density functions separately since they cannot be obtained from each other. The two functions can be related in the limit of large values of the argument of the modified Bessel functions (ref. 22) as follows:

$$\left.\begin{aligned} \frac{N(x;n)}{p_x(x;n)} &\approx \sqrt{2\pi}\, N_o \frac{b|x - x_o|}{A} \\ \frac{|x - x_o|}{Ab} &>> 1 \end{aligned}\right\} \tag{84}$$

Examination of some measured atmospheric turbulence data (ref. 32) which were analyzed by both types of functions shows a qualitative agreement with equations (84).

Another example of the dependence property of the Press process is the associated vector process. In atmospheric turbulence applications the vector process is assumed to be formed by the product of a vector R process and a scalar S process, that is, the vector-scalar form of the process. The components of the Gaussian R process are independent if they are uncorrelated, as is assumed for isotropic turbulence. The components of the resulting Press process are also uncorrelated. However, they are not independent. This property does not affect the joint moments of second order between different components of the turbulence, but it does affect the joint moments of higher order. If the vector-vector form of the Press process is used, then the components of the resulting process can be independent.

The exceedance ratio and the probability distribution function have been used as measures of the rate or probability of failure of a system. For linear systems these measures can be related to a single intensity factor because of the specific functional form of the exceedance ratio (eqs. (82) and (83)) and the probability distribution function (from eq. (81)). These are functions solely of the ratio of the incremental response level and the standard deviation of the response R process for a given set of atmospheric parameters

$$U_\sigma = \frac{|x_l - x_o|}{A} \tag{85}$$

Thus, this turbulence or gust intensity parameter is directly related to a value of the probability of exceedance for a linear system. Also, the gust intensity parameter and the expected number of zero crossings are directly related to a value of the expected number of exceedances of a particular load level. It has been the basis of the development of structural strength criteria for aircraft response to atmospheric turbulence (refs. 14 to 16).

Effect of Intensity Distribution Upon Calculated Exceedances

The effects of the distribution of the intensity (S) component of the product process upon the calculated exceedances of aircraft response and loads are examined in this section. The cases of a Rayleigh and of one other distribution of the intensity process are considered. The development is divided into two parts. First, methods of determining the atmospheric parameters for different intensity distributions are developed. Second, the effects of the distribution of the intensity process upon calculated exceedances are examined. The exceedances are examined by both the mission analysis and the design envelope techniques (ref. 14) which are used to define the required limit load level. The mission analysis technique is applied to the Boeing 720B for which a complete set of mission data is available in reference 15.

The generalized Press process is used to consider different distributions for the intensity process. Two specific cases are considered. The first is a Rayleigh distribution, which is a generalized process of index one. The Rayleigh distribution of the intensity process has been suggested by some examinations of atmospheric turbulence data (for example, refs. 11 and 33). The second is a higher order distribution, of index four, which shows the effects of a more radical change in the distribution of the intensity process. Both cases are compared with the Press process which has a Gaussian distribution of intensity and which is a generalized process of index one-half. Only the fully developed form of the random processes is considered since the analysis is concerned with exceedances measured over a long period of time. The present section uses the terminology of the aeronautical literature; the S component is called the intensity process, whose classification is based upon one type of turbulence only.

The use of the generalized process requires the determination of the parameters of the intensity process (the P's and b's) for different indices of the process. Different sets of the parameters are needed since different indices of the generalized process have different exceedance expressions (eq. (83)). The historical approach is to determine the parameters so that the calculated exceedances match the measured exceedance data. A slightly different approach is used in the present study. A method is developed which modifies the available values of the parameters which are defined for the Press process. This approach is adequate to determine the general effects of the intensity distribution. This approach has two advantages. First, it avoids a reexamination of the basic experimental data. Second, it shows the effect of the intensity distribution on a comparative basis and avoids some open questions on the values of the original parameters for the Press process. Thus, any revision of the original parameters, such as those suggested in references 12 and 34, will have a minor effect on the present comparison.

The present approach for determination of the atmospheric parameters is based upon the idea of matching measured exceedance data. However, the matching of exceedance expressions does not imply the matching of other quantities as the index of the generalized process is changed. One study (ref. 16) shows the impossibility of matching both the calculated exceedances and the variance for different indices of the generalized process. The present approach is based upon matching the exceedance expressions since these represent the basic experimental data used to determine the original parameters.

The parameters for the generalized process are determined by the following procedure. The original parameters of the Press process are scaled by two factors. The same factors are used for all altitudes. The original set of parameters is that of references 14 and 15. The same set of parameters is used in aircraft strength criteria (ref. 18 and (with minor revision) ref. 17). The scaling factors are determined by matching the two exceedance curves in the range of three to four times the standard deviation of the original Press process. As a check, the new parameters are then used in a mission analysis of the Boeing 720B load factor (vertical acceleration of the center of gravity). The calculated exceedances are compared with those for the Press process to see whether they match in the range of 0.6g to 0.8g, which is the primary range of the measured data (refs. 12 and 27). The procedure considers only the storm ($i = 2$) terms. The same factors are applied to the nonstorm ($i = 1$) terms, although these can usually be ignored at the limit load levels of interest.

Thus, the calculated exceedances of the generalized process must match those of the Press process if the parameters are properly defined. This does not say, however, that the two sets of calculated exceedances are identical. Differences can occur for two basic reasons. First, differences can occur at different response levels such as those associated with limit load levels which may involve an extrapolation from the range of the measured data. Second, differences can occur for quantities other than the load factor which is the basis of the measured data.

One approach to the development of structural criteria is the mission analysis technique. This approach is based upon the exceedances in average aircraft usage. The aircraft usage is modeled by a set of representative flight segments. The total exceedances are the sum of the exceedances calculated for each segment, weighted by usage factors. A discussion of this technique is given in reference 14.

The mission analysis technique is applied to the Boeing 720B. A mission analysis of the 720B, using the Press process, was part of the FAA study to develop the structural criteria. (See refs. 14 and 15.) The required response and mission data are given in reference 15. Three response quantities are examined. First, the load factor is examined primarily to check the adjustment of the atmospheric parameters. The second quantity is the critical load for the vertical (symmetric) analysis, wing bending moment at wing eta station 0.33. The third quantity is the critical load for the lateral analysis, vertical-tail bending moment at elastic axis station 158. For the last quantity the yaw damper is assumed to be fully operative by following the procedure of reference 15. In the development of structural criteria based on the mission analysis technique, an exceedance level of 2.0×10^{-5} exceedances per average hour is often used to specify the required level of limit load (refs. 14 and 17).

The first case considered is a generalized process of index one, which has a Rayleigh distribution of the intensity process. The atmospheric parameters are adjusted by the procedure indicated previously

$$P_i(n = 1) = 0.59P_i(n = 1/2)$$

$$b_i(n = 1) = 0.89b_i(n = 1/2)$$

$$E\left[x^2(t;n = 1)\right] = 0.935E\left[x^2(t;n = 1/2)\right]$$

The adjustment factors for the probability and intensity parameters and the resulting factor for the variance, using equations (B2), are indicated. The largest change is in the probability parameter. The changes in the intensity parameter and in the variance of the total process are smaller.

The results of the mission analysis for the generalized process of index one are shown in figures 6 to 8. The results are the exceedances of load level per hour of average airplane usage plotted against the load level. The exceedances for the load factor (fig. 6) show close agreement for the two processes. This agreement indicates that the atmospheric parameters have been adjusted in a reasonable manner. The exceedances for the two loads (figs. 7 and 8) show essentially the same comparison with small differences at limit load levels. In all three cases the Press process gives a conservative specification of limit load; that is, it gives the highest values for limit load level.

The second case considered is a generalized process of index four. The adjustment factors and the resulting factor for the variance, using equations (B2), of the process are as follows:

$$P_i(n = 4) = 0.25P_i(n = 1/2)$$

$$b_i(n = 4) = 0.60b_i(n = 1/2)$$

$$E\left[x^2(t;n = 4)\right] = 0.72E\left[x^2(t;n = 1/2)\right]$$

All the quantities are changed significantly. The largest change is again in the probability parameter.

The results of the mission analysis for the generalized process of index four are shown in figures 6 to 8. The exceedances of the load factor (fig. 6) show reasonable agreement in the range of 0.6g to 0.8g and indicate a reasonable determination of the atmospheric parameters. However, significant differences occur at higher response levels. For all three response quantities the differences at limit load levels are around 10 percent. In all three cases the Press process gives a conservative specification of limit load.

A second approach to the development of structural criteria is the design envelope technique. This criteria is based upon a single flight condition. The criteria is the intensity factor (eq. (85)) which implicitly specifies an associated exceedance ratio. Figure 9 shows the value of the intensity factor as a function of altitude for specific values of the exceedance ratio (eq. (83)) and for the three generalized processes under consideration. This method follows the procedure of reference 14. For reference, a value of 1.2×10^{-6} is suggested for the criteria exceedance ratio in reference 14. The values of figure 9 indicate the effects of the distribution of the intensity process upon the intensity factor U_σ. For the generalized process of index one, the differences from the Press process are insignificant. For the generalized process of index four, the differences are small. In both cases the Press process gives a conservative specification of limit loads. It is noted that there is a modification of this approach that uses the exceedances rather than the exceedance ratio to specify limit load levels (ref. 16). The present conclusions also apply to that approach since the only difference is the expected number of zero crossings of the R process which is independent of the intensity process.

The preceding discussion shows that the calculated exceedances are largely independent of the distribution of the intensity process unless data are available over a wide range of the response. Thus, it is generally difficult to determine the distribution of the intensity process from measured exceedance data. A better approach is the use of the flatness factor, which is more sensitive to the distribution of the intensity process. If the measured data represent only one type of turbulence, the flatness factor is directly related to the index of the generalized process. For a fully developed process, the flatness factor has a simple form

$$M_4 = \frac{3(n + 1)}{n} \tag{86}$$

In summary, the effects of the distribution of the intensity process upon calculated exceedances are examined by using the generalized process. The exceedances are compared with those of the Press process which has a Gaussian intensity distribution. Several conclusions are made. First, it is necessary to redefine the atmospheric parameters (P's and b's) as the intensity process is changed. The parameters can be defined so that the calculated exceedances reasonably match the measured data. Second, the use of different distributions for the intensity process has a minor effect on the limit loads determined by the mission analysis technique. For a Rayleigh intensity distribution the effect is negligible. For higher index distributions the differences may not be negligible. However, the Press process, with a Gaussian intensity distribution, gives a conservative specification of limit load levels. Finally, it is generally difficult to determine the distribution of the intensity process from measured exceedance data. A better method is the examination of flatness factors to determine the corresponding intensity distribution.

CONCLUDING REMARKS

The present report is an examination of the random process, termed the Press process, which is used to model atmospheric turbulence in aircraft response problems. The primary conclusion of the present study is that the basic concepts of the Press process, which were originally developed largely on an intuitive basis, can be derived from an explicit mathematical model. The model is based upon the product of two independent random processes, which have radically different time scales. With this model it is possible to describe explicitly the general properties of the Press process such as the locally Gaussian condition, the Gaussian and non-Gaussian forms of experimental data, and the possible nonstationarity of experimental exceedance data.

The concept of the product process introduces two independent component processes. One component is the R process, always specified as Gaussian, which defines the variation of the product process at constant intensity or standard deviation. The other component is the S process which has a slower variation. The first, second, and higher order probability density and characteristic functions are obtained for the product process.

In developing these relations, both the usual Gaussian process and a general class of processes, related to the Pearson type III family, are considered for the S component process.

The concept of a product process leads to an approximate procedure for the analysis of the response of linear dynamic systems to a class of non-Gaussian random processes. The procedure is based upon the concept that the time variation of the S component process is much slower than that of both the R component process and all significant aspects of the system dynamics. Under this quasi-steady assumption the sytem dynamics are influenced only by the R process, which is Gaussian. The variation of the S component process does not affect the system dynamics, but does introduce the non-Gaussian nature of the random process in a static manner.

In the mathematical model the random process develops in two stages since the process is the product of two independent processes which develop on radically different time scales. The second stage of development is of primary interest, the development of the slower S component process with the R component process being fully developed. The mathematical model explicitly shows the transition from the Gaussian form of the R component process to the non-Gaussian form of the fully developed Press process. This transition, which is nonstationary, is explicitly described by the transition properties of the stationary Press process.

The effects of the form of the distribution of the S component process upon calculated exceedances are examined. The cases of a Rayleigh and of a second distribution are compared with the Gaussian case. It is concluded that the calculated exceedances are not greatly influenced by the distribution of the S component if the atmospheric parameters are properly defined. A specific numerical example of a mission analysis is computed for one airplane. The limit load levels determined by the criteria specification are compared with those computed from the Press process, with a Gaussian S component process. The changes due to the Rayleigh distribution for the S process are negligible. The use of the second process causes some differences at limit load levels. However, in the cases considered, comparison with the Press model shows that the differences are either negligible or that the Press model predicts somewhat higher occurrence of limit loads; that is, the Press model is conservative.

The present study outlines the analytical properties of the Press process. Several applications of the formulation to the measured properties of atmospheric turbulence and several extensions of the analysis are suggested.

Several properties of measured turbulence data which have not been examined on an explicit basis are identified. The primary areas of interest are the transition properties and the associated nonstationary process. The Press model predicts explicit relations for the transition properties of the exceedances, probability density function, moments, flatness factor, etc. This prediction raises the basic question of the correspondence between the predicted properties and the measured properties of atmospheric turbulence: does the analytical model adequately match the nonstationary properties of the measured data? An associated question is the specific form of the time variation of the S component process: what are the power spectral or covariance functions of the S process?

The present study suggests two problems of a mathematical nature. First, the use of the transition properties of a stationary product process presents a method for the analysis of nonstationary random processes. Some of the properties of the associated nonstationary process are outlined in the present study. However, the relation between the present approach and other approaches to the analysis of nonstationary processes presents an interesting mathematical study. Second, the concepts of the product process and of the quasi-steady approximation present a method of analyzing the response of linear dynamic systems to a class of non-Gaussian random processes. The justification for the approximation is discussed largely on a qualitative basis in the present study. This procedure should be thoroughly examined as a general analytical method.

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APPENDIX A

DEVELOPMENT OF Σ PROCESS

There are two random processes associated with the S component of the Press process. One is the Σ process, which is associated with the historical development of the Press model. The other is the S process itself, which is required for a complete mathematical formulation. The relations between these two processes are developed in the present section.

The difference between the two random processes is the range of their values: the Σ process is restricted to positive values whereas the S process has both positive and negative values. The values of the Σ process are the moduli of the values of the S process

$$\sigma = |s| \tag{A1}$$

The Σ process appears when the Gaussian R component process is related to a conditional process by using equations (4) and (11),

$$p_c(z|s) = \frac{1}{\sqrt{2\pi} A\sigma} \exp\left(\frac{-z^2}{2A^2\sigma^2}\right) \tag{A2}$$

The conditional process is Gaussian with standard deviation equal to $A\sigma$ which must be positive. Thus, the historical development, using the idea of a conditional process through the locally Gaussian condition, is based on the Σ process. Also, the analysis of turbulence data in terms of its standard deviation leads to the Σ rather than the S process.

The properties of the Σ process are determined from the specified properties of the S process by equation (A1). The properties are obtained either from the probability density function of the Σ process or directly from the properties of the S process. For example, the moments of the Σ process are the absolute moments of the S process. The relations between the dynamic properties of the two processes are more complicated. The quantity of primary interest is either the autocovariance or the autocorrelation function of the Σ process

$$E\left[\sigma(t_1)\ \sigma(t_2)\right] = b^2\ \rho_\sigma(\tau) = E\left[\left|s(t_1)\ s(t_2)\right|\right] \tag{A3}$$

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For the case of the Gaussian S component process, the relation between the autocorrelation functions is (ref. 19)

$$\left.\begin{aligned} \rho_{\sigma}(\tau) &= \rho_{s}(\tau) + \frac{2}{\pi}(\sin\gamma - \gamma\cos\gamma) \\ \cos\gamma &= \rho_{s}(\tau) \end{aligned}\right\} \tag{A4}$$

The dynamic properties of the two processes are determined from their autocorrelation functions. The specific relations are obtained from equations (A4) either by evaluating the derivatives of the autocorrelation functions or by expanding the functions in power series in the time difference variable. These procedures give the following relations:

$$\left.\begin{aligned} \rho'_{\sigma}(0) &= 0 \\ \rho''_{\sigma}(0) &= \rho''_{s}(0) \\ \rho'''_{\sigma}(0+) &= \frac{-4}{\pi}\left[-\rho''_{s}(0)\right]^{3/2} \end{aligned}\right\} \tag{A5}$$

Thus, the variances of the first derivatives of the S and Σ process are equal. However, the Σ process does not possess a second derivative since the corresponding variance has an infinite value. This is the primary reason the S process must be used in the formulation of the Press process: its dynamic properties can be specified arbitrarily. Thus, the quasi-steady approximation cannot be developed if the Σ process is used, since the second derivative of that process does not exist. However, once the quasi-steady approximation is established, the Σ process can be used in place of the S process since only the first order properties are used.

In the aeronautical literature the Σ process is often referred to as the "intensity" of the turbulence. This interpretation is based upon the standard deviation of the conditional process (eq. (A2)) being equal to $A\sigma$. Strictly speaking, the term is a misnomer since an intensity is related to the variance and not the standard deviation of a random process. However, the term intensity is commonly used to identify the Σ process. The S process is also identified as the intensity, if it is remembered that the standard deviation of the conditional process depends on the modulus of the S process.

In summary, both the S and Σ processes are important aspects of the associated component of the Press process. The S process, with both positive and negative values, must be used in the mathematical formulation. The Σ process, restricted to positive values, is required for the analysis of turbulence data. Also, the properties of the S process cannot be completely determined from turbulence data through the associated conditional process (eq. (A2)). The properties of the Σ process can be developed from the specified properties of the S process by using the transformation of equation (A1).

APPENDIX B

DEVELOPMENT OF GENERALIZED PRESS PROCESS

The generalized Press process is formed by the product of the Gaussian R process and the S process, which is specified as a member of a family related to the Pearson type III family of processes. The probability density functions of the S process family are given in equation (22). The moments and both the probability density and characteristic functions of the corresponding generalized processes are developed in this appendix. The relation between generalized processes of various indices is also developed.

First Order Distributions

The variance of the scalar S process is determined from its probability density function (eq. (22))

$$E\left[s^2(t_1;n)\right] = 2n\, b^2(n) = 2nb^2 \tag{B1}$$

The parameter b is not equal to the standard deviation of the S process in the general case. Also, this parameter can have different values for different generalized processes, that is, for different values of the index n, although the specific notation is dropped in the subsequent discussion.

The moments of the scalar product process are determined by the moments of the R and S component processes and by the product relation (eq. (1)). (The generalized process is denoted as the Y process to distinguish it from the Press process itself.)

$$\left.\begin{aligned}
E\left[y^\ell(t_1;n)\right] &= E\left[r^\ell(t_1)\right]E\left[s^\ell(t_1;n)\right] \\
E\left[y^\ell(t_1;n)\right] &= \begin{cases} 0 & (\ell = \text{odd}) \\ \dfrac{2^\ell}{\sqrt{\pi}}\,\dfrac{\Gamma\left(n + \frac{\ell}{2}\right)}{\Gamma(n)}\,\Gamma\left(\dfrac{\ell + 1}{2}\right)A^\ell b^\ell & (\ell = \text{even}) \end{cases}
\end{aligned}\right\} \tag{B2}$$

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The probability density and characteristic functions of the scalar generalized process follow from the explicit forms of the functions of the R and S processes and from application of equations (3) and (6)

$$p_y(y;n) = \sqrt{\frac{2}{\pi}}\,\frac{1}{\Gamma(n)}\,\frac{1}{bA}\left(\frac{|y|}{bA}\right)^{n-\frac{1}{2}} K_{n-\frac{1}{2}}\left(\frac{|y|}{bA}\right) \tag{B3}$$

$$C_y(\theta;n) = \left(b^2A^2\theta^2 + 1\right)^{-n} \tag{B4}$$

The formulation can be extended to vector processes without difficulty. The structure of the vector-scalar process is expressed most concisely by the characteristic function, which has a functional form similar to that of equation (19)

$$C_y(\overline{\theta};n) = \left(b^2 \sum_{i=1}^{m} \sum_{j=1}^{m} \theta_i H_{ij} \theta_j + 1\right)^{-n} \tag{B5}$$

The generalized Press process has a strongly non-Gaussian character, as indicated by the exponential-type dependence of the probability density functions for extreme values of the process. Another indication is the associated flatness factor

$$M_4 = \frac{E\left[y^4(t_1;n)\right]}{\left\{E\left[y^2(t_1;n)\right]\right\}^2} = \frac{3(n + 1)}{n} \tag{B6}$$

The values of the flatness factor are larger than the value of three for a Gaussian process. However, the flatness factor does approach the Gaussian value of three in the limit of large values of the index n.

The index of the generalized process appears only as a power coefficient in the characteristic function (eq. (B5)). This property indicates a relation between the generalized processes of various indices, since the characteristic function of a sum of independent random variables is the product of the individual characteristic functions. The characteristic functions of the generalized process (eq. (B4)) and the Press process (eq. (13)) have a simple functional relation

$$C_y(\theta;n) = \left[C_y(\theta;n = 1/2)\right]^{2n} = \left[C_z(\theta)\right]^{2n} \tag{B7}$$

Thus, the sum of 2n independent and identically distributed Press processes is a generalized process of index equal to n

$$y(t_1;n) = \sum_{j=1}^{2n} z_j(t_1) \tag{B8}$$

This relation presents an alternate method for developing the properties of the generalized process. For example, the relations for the moments (eqs. (B1) and (B2)) can be developed by this method.

The development suggests a relationship to the central limit theorem. Interpreting the generalized process of index n as the sum of 2n independent and identically distributed Press processes, the central limit theorem requires that the generalized process becomes Gaussian in the limit of large n. The validity of this statement can be shown by examination of the characteristic function (eq. (B4)). This conclusion explains why the flatness factor (eq. (B6)) approaches the Gaussian value of three in the limit of large n.

Second Order Distributions

The development of second and higher order distributions of the generalized Press process requires specification of the higher order distributions of the S process. For the Press process, the S process is stationary and Gaussian; the higher order distributions are specified directly. For the generalized process it is simpler to use an alternate approach.

For the first order distributions, the generalized Press process can be developed from the Press process itself by equation (B8). This relation is extended to the case of random processes considered at multiple time points

$$y(t;n) = \sum_{j=1}^{2n} z_j(t) \tag{B9}$$

The random process Y is a generalized Press process of index n if the Z_j processes are all Press processes (generalized processes of index one-half), which are independent and identically distributed. By using the relationship between the generalized process and the Press process (eq. (B9)), the second order characteristic functions are developed

$$C_y(\theta_1,\theta_2;n) = \left[C_z(\theta_1,\theta_2)\right]^{2n} \tag{B10}$$

This relation is similar to equation (B7) for the first order distribution. The second order characteristic function can be used to develop relations for the first derivative, the joint distribution of the process and its first derivative, and the exceedance expression. Thus, the second and higher order distributions for the generalized processes of various indices can be developed directly from the corresponding distributions of the Press process.

APPENDIX C

COMPARISON OF THE PRODUCT AND OSCILLATORY RANDOM PROCESSES

The relation between the definition of the product process and that of the oscillatory process developed by Priestley (ref. 29) is examined in this appendix. It is shown that the spectral expansions of two random processes have a formal similarity. The discussion is restricted to the spectral expansions; the probabilistic structures of the random processes are not considered.

The time variation of a random process is described by its spectral expansion. The product random process and its two component processes are stationary and are assumed to be mean square continuous. Accordingly, they can be expressed in a Fourier or spectral expansion (refs. 25 and 35)

$$r(t) = \int_{-\infty}^{\infty} e^{i\omega t}\, dR(\omega) \tag{C1}$$

The process $\{dR(\omega)\}$ is the spectral process associated with the R process. The spectral process, having orthogonal increments, is the basis of the definition of the spectral distribution and power spectral density functions of the original process.

$$E\left[\left|dR(\omega)\right|^2\right] = dF_r(\omega) = \Phi_r(\omega)\, d\omega \tag{C2}$$

Similar relations can be written for the S and Z processes. By using the product relation, the power spectral density functions of the three processes can be related and will lead to equation (30).

The Fourier expansion of the product process offers some insight into the time evolution of the process and into the relationship with the concept of a nonstationary process. By using the interpretation of the S process as slowly varying in time, the spectral expansion of the product process is

$$z(t) = r(t)\, s(t) = \int_{-\infty}^{\infty} e^{i\omega t}\, s(t)\, dR(\omega) \tag{C3}$$

This relation follows directly from the product relation (eq. (23)) and the Fourier expansion of the R process (eq. (C1)).

A similar relation is written for the Fourier expansion of a nonstationary process by following the concepts of evolutionary spectra and oscillatory processes developed by Priestley (ref. 29)

$$z(t) = \int_{-\infty}^{\infty} e^{i\omega t} a_o(t)\ dZ(\omega) \tag{C4}$$

This relation is the spectral expansion of an oscillatory process in the special case of a uniformly modulated process. The modulation function $a_o(t)$ is a slowly varying deterministic function.

The spectral expansions of the oscillatory process (eq. (C4)) and the product process (eq. (C3)) have a strong formal similarity. In both cases a spectral process is modulated by a time function which is considered to be slowly varying relative to the unmodulated process. With the oscillatory process the modulation function is a deterministic function; the resulting oscillatory random process is nonstationary. With the product process the modulation function is a random process; the resulting product random process is stationary. The stationarity of the product process is determined by the functional properties of the probability density and characteristic functions and by the definition of stationarity. The nonstationary aspect of the product process requires an examination of the transition properties, which are considered in the main text.

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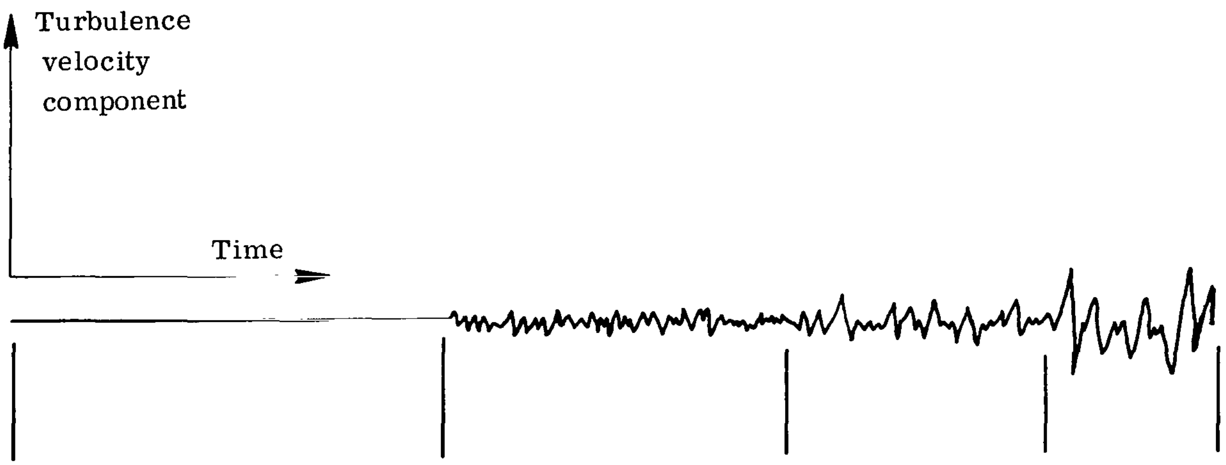

Figure 1.- Atmospheric turbulence model with locally Gaussian regions and intensity variations.

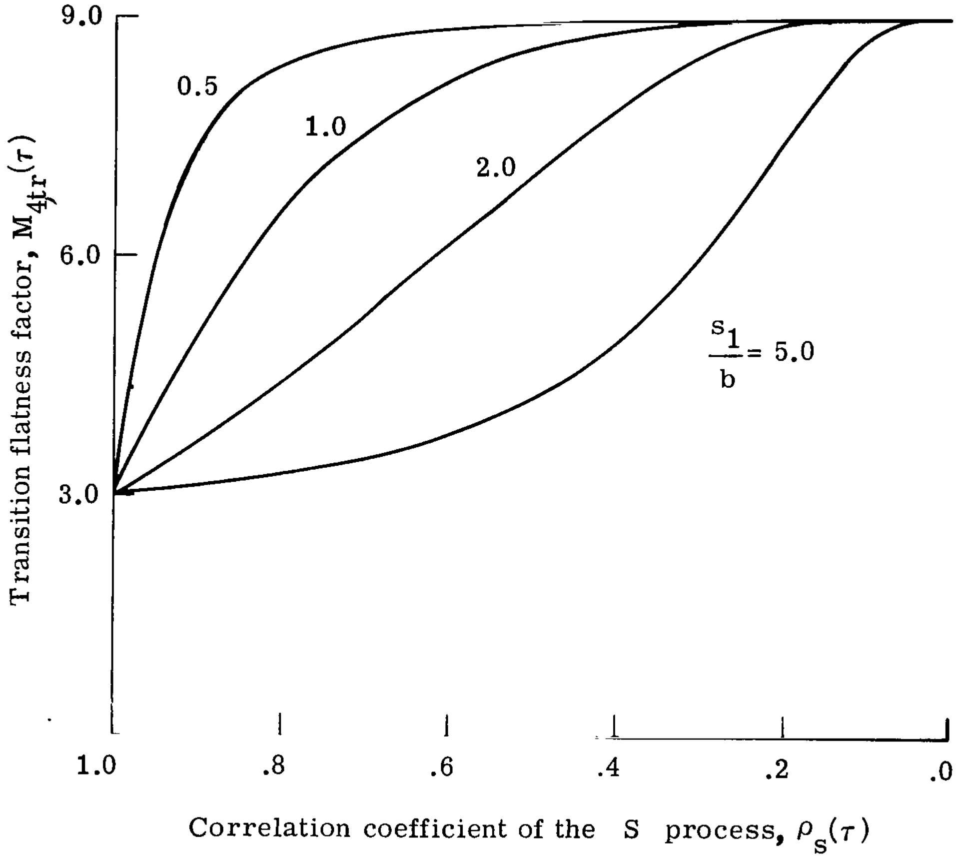

Figure 2.- Transition flatness factor for the Press process.

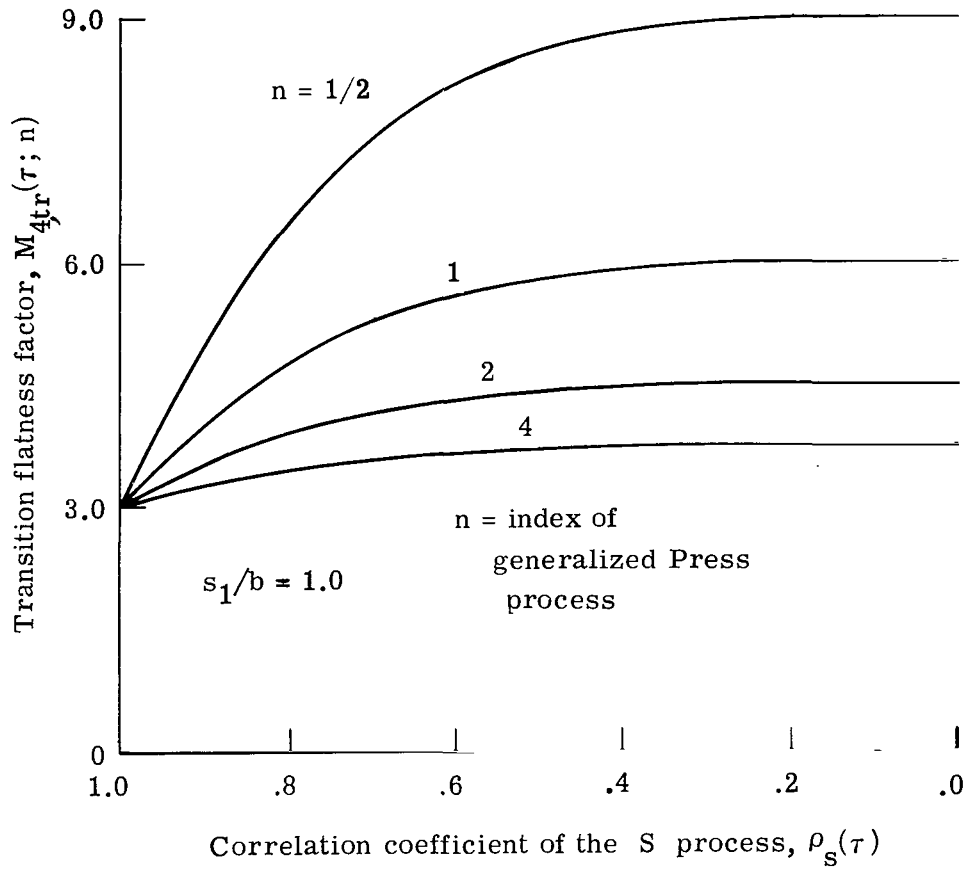

Figure 3.- Transition flatness factor for generalized Press process.

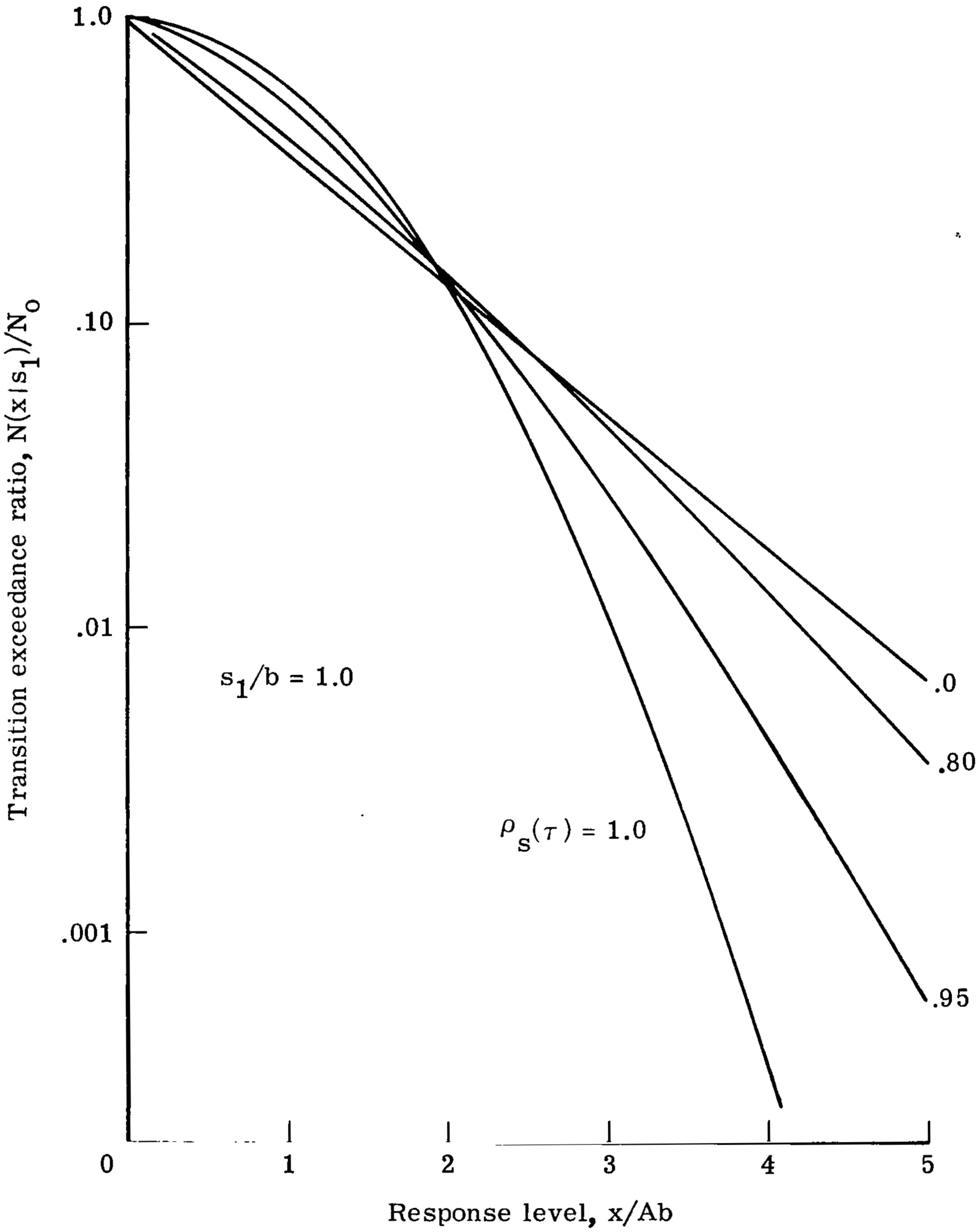

Figure 4.- Transition exceedance ratio for Press process. $s_1/b = 1.0$.

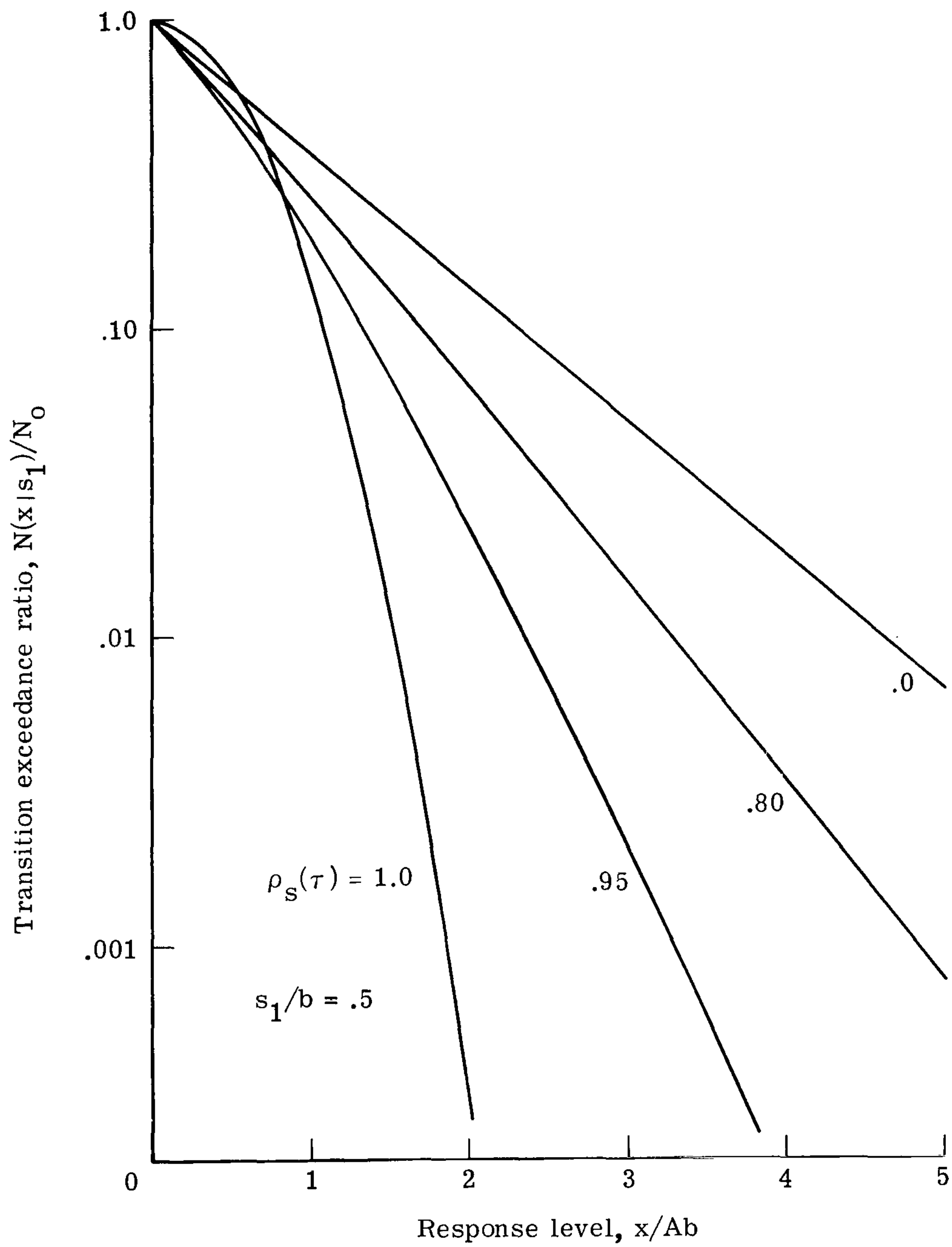

Figure 5.- Transition exceedance ratio for Press process. $s_1/b = 0.5$.

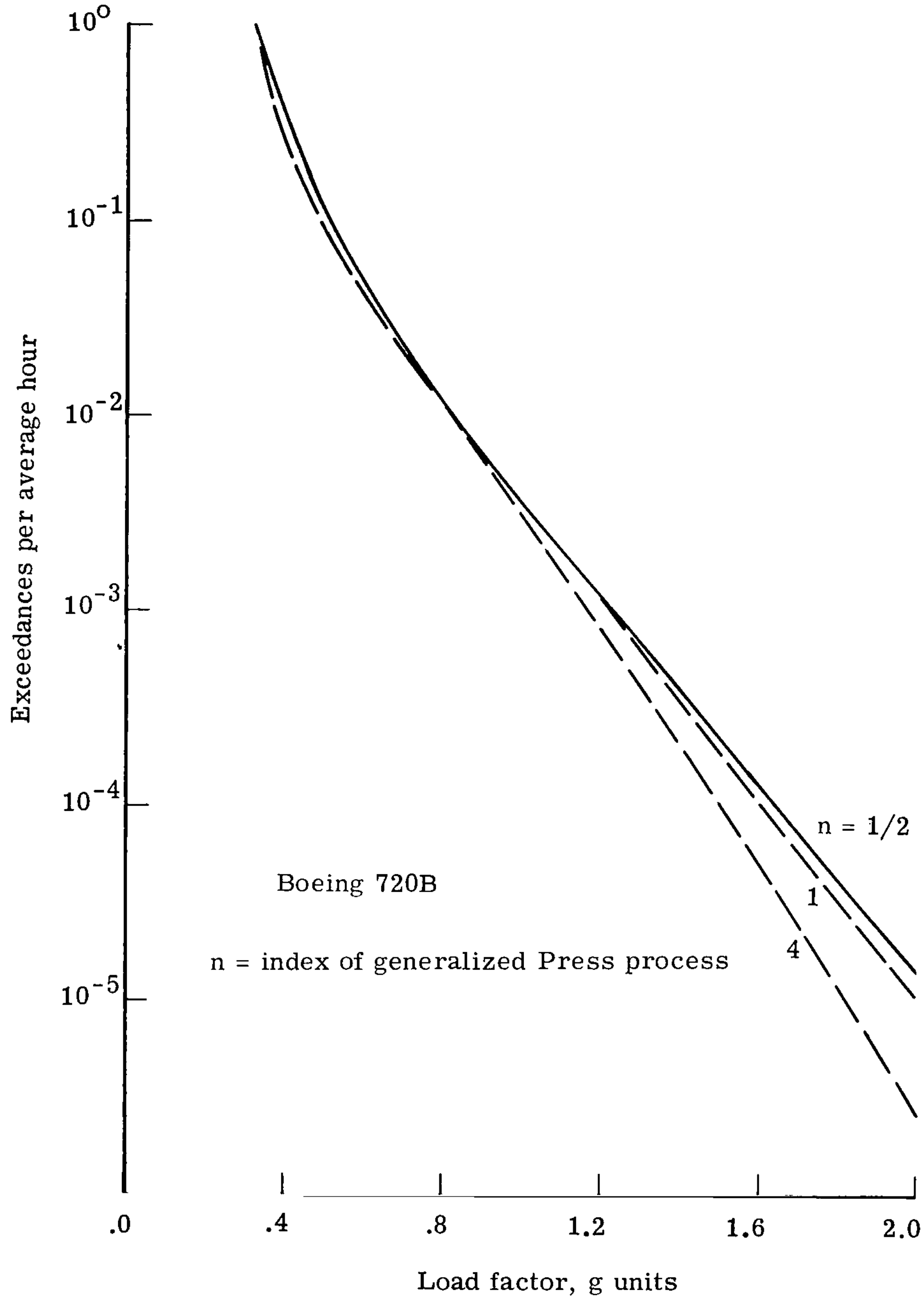

Figure 6.- Mission analysis for load factor, generalized Press process.

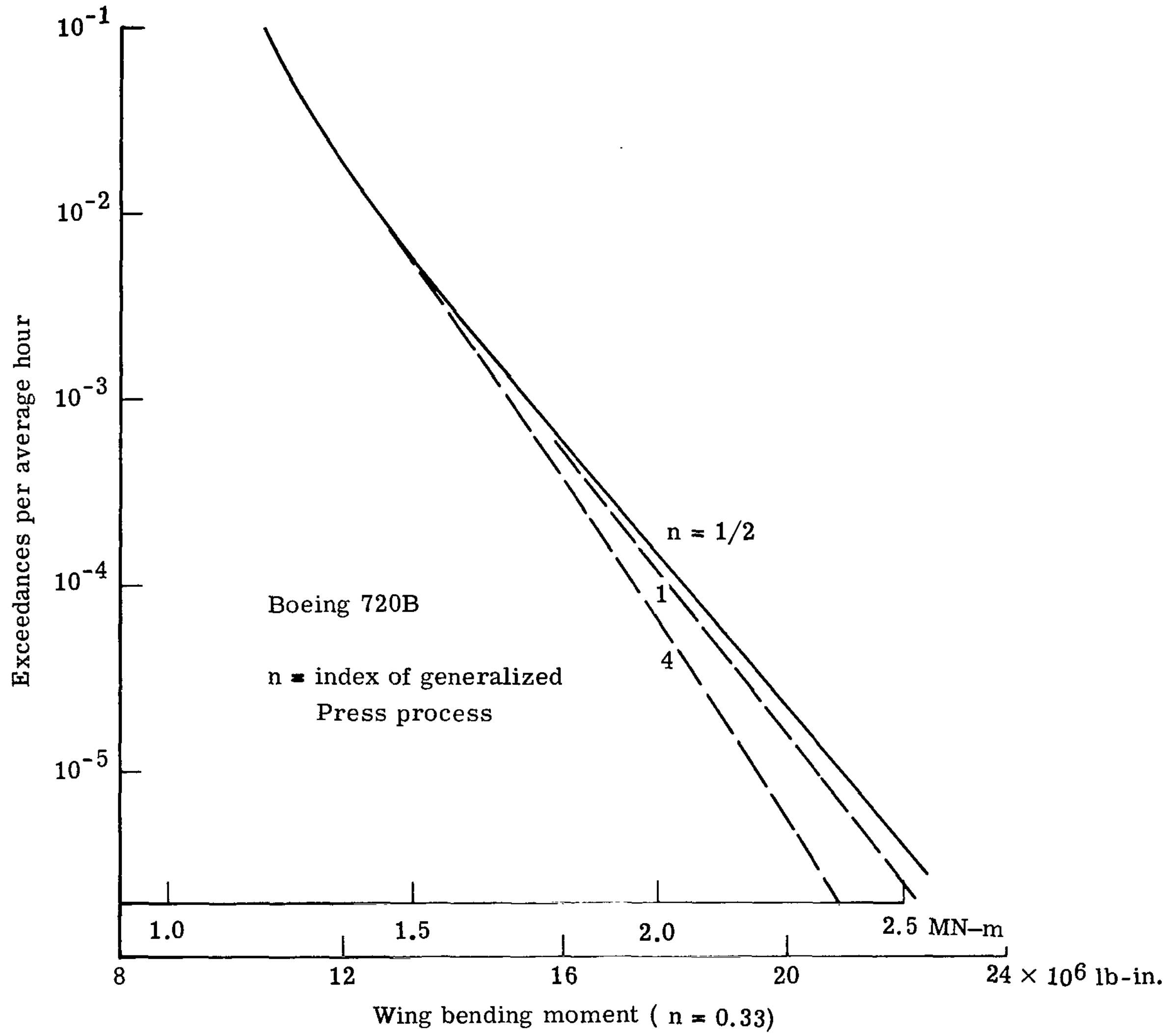

Figure 7.- Mission analysis for wing bending moment, generalized Press process.

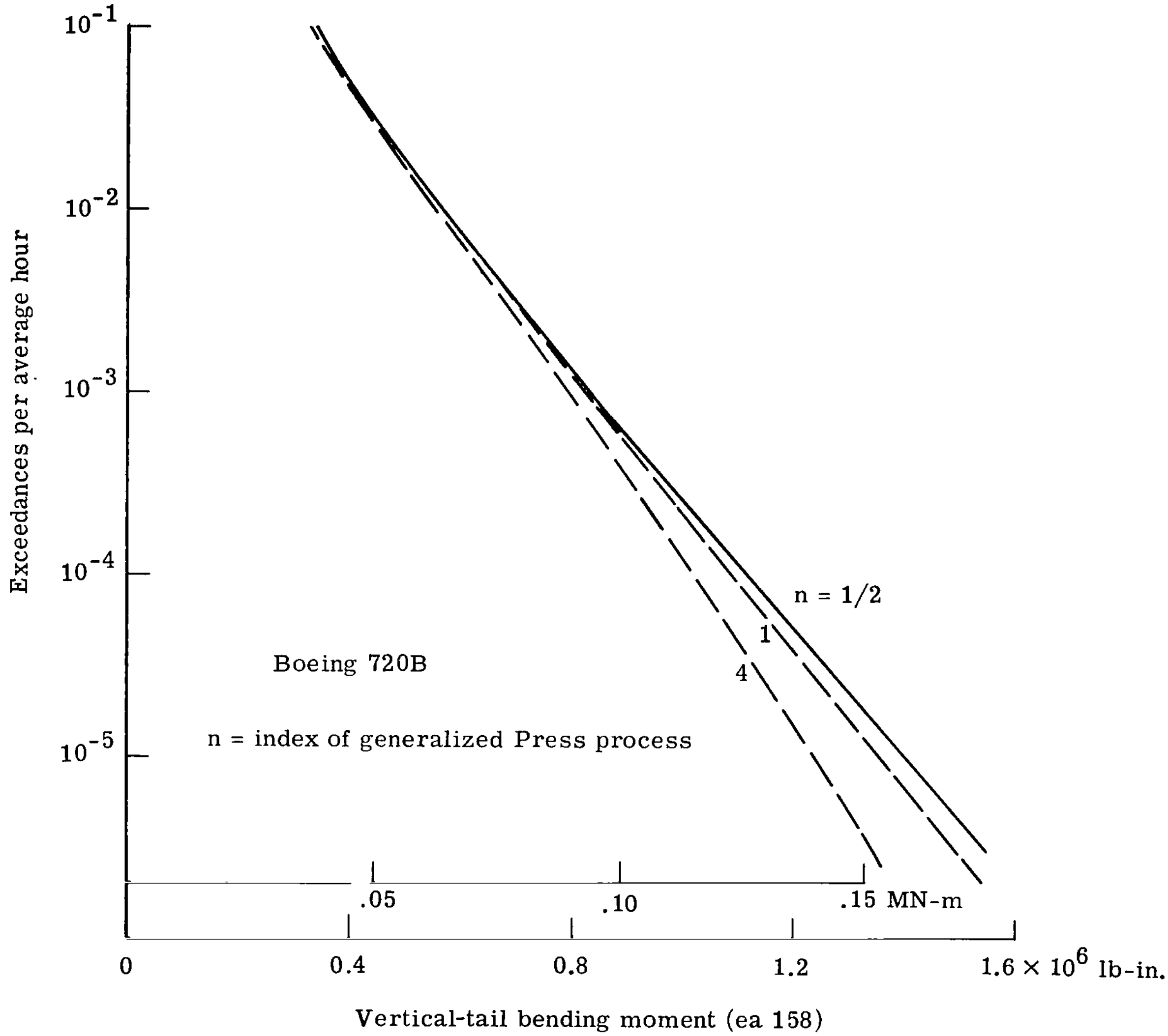

Figure 8.- Mission analysis for vertical-tail bending moment, generalized Press process.

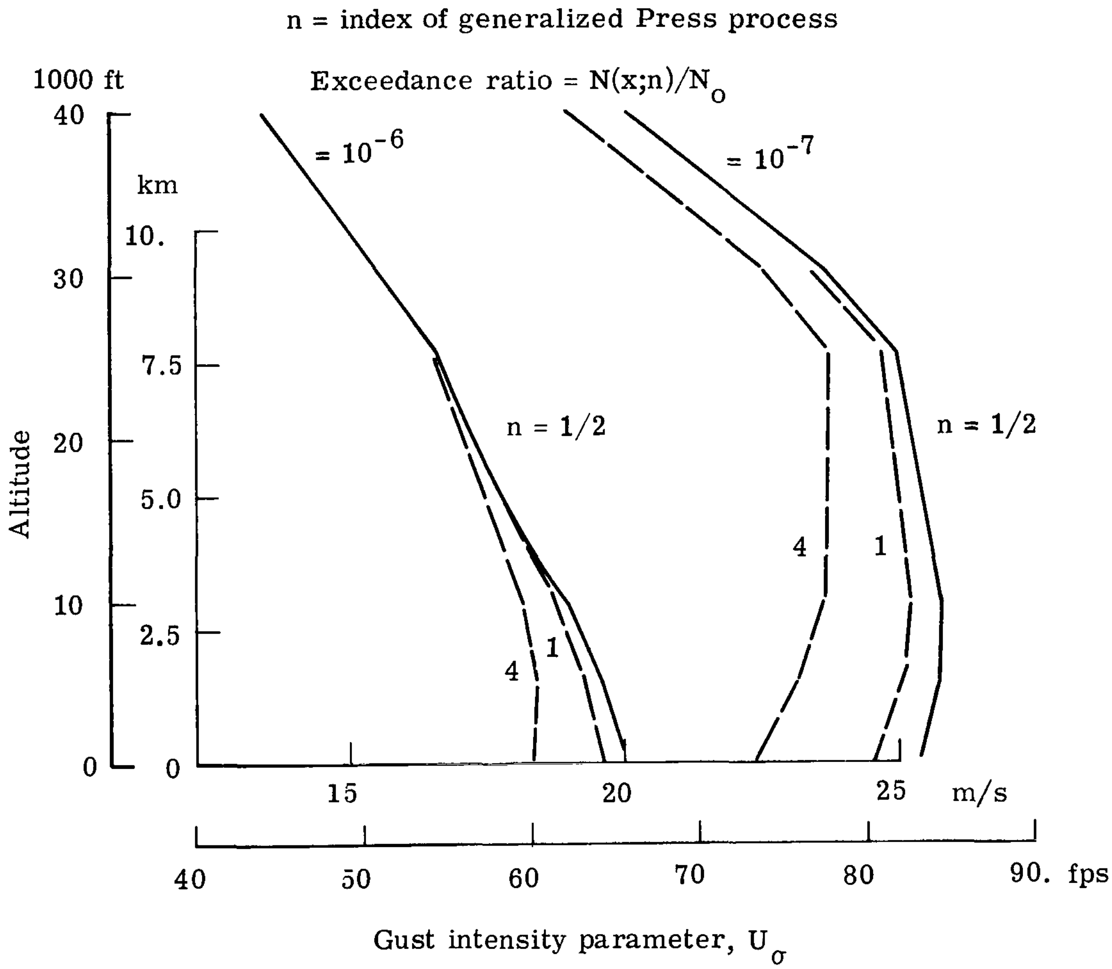

Figure 9.- Effect of intensity distribution upon variation of U_σ with altitude.

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